

C of C Math Meet 2006 MARATHON

Rules:

- i. The problems are to be worked out individually and independently. Only textbooks and library sources may be used. Calculators and computers may be used. Each entry must be signed by a math teacher within the school to certify that all rules have been followed. Any number of entries from a school may be submitted.
- ii. Work must be shown neatly and concisely. Explain how you got your answer. It is possible that several entries will have correct solutions, so work will be judged on exposition, clarity of thought and ingenuity, as well as correctness. The date of submission will also be considered. Electronic submissions will be accepted only once.
- iii. All entrants must be students who have not graduated from high school. All entrants must be registered for the Math Meet.
- iv. The judges' decisions will be final.
- v. All papers are to be mailed to the following address or submitted electronically to mathmeet@cofc.edu

Math Meet
MATH MARATHON
Department of Mathematics
College of Charleston
Charleston, SC 29424

- vi. The cover paper for each entry must have the following information: (This may be turned in the day of the Math Meet if submitted electronically and not mailed.) Student Name, Math Marathon, Home Address, E-mail Address, School; Year of Graduation, School Address, Signature of a Math Teacher for Verification .
- vii. All entries must be received or postmarked by February 14, 2006.

1. Find the smallest pair of consecutive integers whose product is a multiple of 2006.
2. A figure consists of two circles, each of which pass through the center of the other. The smallest region formed has area equal to one square unit. Find the radii of the circles.

3. The function

$$f(x) = \frac{ax + b}{cx + d}$$

is its own inverse. It also has exactly two fixed points, whose sum is $f(2006)$ and whose product is $f(0)$. Determine the constants a, b, c, d . (A *fixed point* of a function f is a value u with the property that $f(u) = u$.)

4. In base 3, base 4, base 5, or base 7, the sum of the digits of this number is 5. In base 2, the number has ten digits. Find the number.
5. A city is surrounded by a wall which has twelve guard towers; the towers are not necessarily evenly spaced around the perimeter. At noon twelve guards begin a twelve hour patrol. Just before noon, each guard randomly decides to begin walking either clockwise or counterclockwise around the wall. Each guard starts at a different tower and begins walking around the wall, always maintaining a pace sufficient to complete exactly one trip around the wall per hour. Whenever two guards meet on patrol, each immediately reverses direction, always maintaining the same pace. Prove that at midnight each guard will be back at the tower where they began.