

MATH MARATHON 2015

Instructions

- The problems are to be worked out individually and independently. Only textbooks and library sources may be used. Calculators and computers may be used. Each entry must be signed by a math teacher within the school to certify that all rules have been followed. Any number of entries from a school may be submitted.
- Work must be shown neatly and concisely. Explain how you got your answer. It is possible that several entries will have correct solutions, so work will be judged on exposition, clarity of thought and ingenuity, as well as correctness. The date of submission will also be considered. Electronic submissions will be accepted only once.
- All entrants must be students who have not graduated from high school. All entrants must be registered for the Math Meet.
- The judges decisions will be final.
- All papers are to be submitted electronically to mathmeet@cofc.edu or mailed to the following address

Math Meet (Marathon)
Department of Mathematics
College of Charleston
66 George Street
Charleston, SC 29424

- The cover paper for each entry must have the following information: (This may be turned in the day of the Math Meet if submitted electronically and not mailed.) Student Name, Math Marathon, Home Address, E-mail Address, School; Year of Graduation, School Address, Signature of a Math Teacher for Verification.
- All entries must be received or postmarked by February 6, 2015.

The Questions

1. For a positive integer n , the positive integer $T_n = \frac{n(n+1)}{2}$ is called the n -th *triangular number*, since a set of T_n objects may be arranged in a triangular array. Find all representations of 2015 as a difference of two triangular numbers.

2. Find an equation of the largest ellipse which is contained in the region

$$R = \{(x, y) : x^2 + y^2 \leq \max\{1, 2|x|\}\}.$$

3. Consider the two rational functions

$$f(x) = 1 - \frac{1}{x}, \quad g(x) = \frac{1}{1-x}.$$

Find all real values of a such that $f^{(2015)}(a) = g^{(2015)}(-a)$, where

$$f^{(n)} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ copies}}$$

denotes the n -fold compositional iterate of f .

4. In a certain game, a coin is tossed repeatedly until it lands heads. If the first heads occurs on the n -th toss, the player gains n dollars if n is odd, but loses n dollars if n is even. What is the expected value of the player's winnings?

5. Find a rational function $G(x)$ with the property that $G(g(x)) = g(7x)$, where

$$g(x) = \frac{e^x - 1}{e^x + 1}.$$