Making Change

All-Day Sprint

College of Charleston Math Meet 2009

How many ways are there to make 10 cents using pennies and nickels?

How many ways are there to make 25 cents using pennies, dimes, and nickels?

How many ways are there to make 50 cents using pennies, dimes, nickels, and quarters?

How many ways are there to make a dollar ($= 100$ cents) using just pennies and dimes?

How many ways are there to make a dollar using just nickels, dimes, and quarters?

How many ways are there to make a dollar using pennies, nickels, dimes, and quarters?

What does making change have to do with Ramanujan?

A partition of a positive integer is a way of writing it as a sum of integers from a particular set. For example, here are all the partitions of 5 over the integers:

- $5 = 5$
- $5 = 1 + 4$
- $5 = 2 + 3$
- $5 = 1 + 1 + 3$
- $5 = 1 + 2 + 2$
- $5 = 1 + 1 + 1 + 2$
- $5 = 1 + 1 + 1 + 1 + 1$

Of these, only three give partitions of 5 over the odd integers, and two give partitions of 5 over the perfect squares.

Srinivasa Ramanujan did a lot of research on the number of ways to partition integers. For example, he and G.H. Hardy estimated that the number of partitions $P(n)$ for large n is approximately

$$P(n) \approx \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}} \text{ as } n \to \infty$$

He also studied generating functions like

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \ldots = 1 + P(1)x + P(2)x^2 + P(3)x^3 + \ldots$$

which are related to modular functions and the mock-θ functions.

The change-making problems are about counting partitions of integers over multiples of 25 (quarters), 10 (dimes), 5 (nickels), and 1 (pennies).