The Area of a Regular Pentagon  
All-Day Sprint / Math Meet 2010

Note: This Sprint should be done without the use of calculators, computers or other references. Use only your brains and your writing implements.

If you’ve studied trigonometry, I bet you know the exact formulas for the sine and cosine of $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{6}$. These are easy enough to figure out by cutting a square in half and by cutting an equilateral triangle in half. But what about something like $\frac{\pi}{5}$? It turns out that with a little help from complex numbers, you can also get an exact formula for the sine and cosine of $\frac{\pi}{5}$. And from that you can find an exact algebraic formula for the area of a pentagon.

First, something you may not yet have seen is that there is a nifty relationship between trig functions, the natural exponential function $e^x$, and complex numbers. Using $i = \sqrt{-1}$, the basic formulas are

$$\cos \omega = \frac{e^{i\omega} + e^{-i\omega}}{2} \quad \sin \omega = \frac{e^{i\omega} - e^{-i\omega}}{2i} \quad e^{i\omega} = \cos \omega + i \sin \omega$$

So for example, you can rediscover the double angle identities like so:

$$\left(e^{i\theta} + e^{-i\theta}\right)^2 = 2^2 (\cos \theta)^2$$

But you can also expand it out

$$\left(e^{i\theta} + e^{-i\theta}\right)^2 = e^{2i\theta} + 2 + e^{-2i\theta} = \left(e^{2i\theta} + e^{-2i\theta}\right) + 2$$

Then use the basic formulas with $\omega = 2\theta$ to get

$$\left(e^{i\theta} + e^{-i\theta}\right)^2 = 2 \cos 2\theta + 2$$

Now set the two calculations equal

$$2^2 (\cos \theta)^2 = 2 \cos 2\theta + 2$$

and you can solve for $\cos 2\theta$ in terms of $\cos \theta$ to get the familiar

$$\cos 2\theta = 2(\cos \theta)^2 - 1$$
* Find a formula for \( \cos 3\theta \) as a polynomial in \( \cos \theta \). That is, find an identity of the form

\[
\cos 3\theta = A_0 + A_1 \cos \theta + A_2 (\cos \theta)^2 + A_3 (\cos \theta)^3
\]

* Find a formula for \( \cos 5\theta \) as a polynomial in \( \cos \theta \).

* Plug \( \theta = \frac{\pi}{10} \) into your \( \cos 5\theta \) identity to form a 5th degree polynomial equation satisfied by the unknown quantity \( x = \cos \frac{\pi}{10} \).

* Solve that polynomial to find an exact formula for \( \cos \frac{\pi}{10} \) in terms of addition, subtraction, multiplication, division, and square roots. Be clever!

* Find an exact formula for the area of a regular pentagon inscribed in a circle of radius 1 in terms of addition, subtraction, multiplication, division, and square roots.