Are these Games Unfair?
All-Day Sprint / Math Meet 2010

Background: Expected Winnings

Suppose you are about to play a game in which you can win or lose money. The way to determine your *expected winnings* is to compute the product of each possible amount you can win with the probability that you will win that amount and then add up all of these products.

For example, if you are playing a game in which there is a probability of 0.2 that you will win $\$10$, a probability of 0.2 that you will win $\$2$ and a probability of 0.6 that you will lose $\$1$, then your expected winnings are:

$$(.2 \times 10) + (.2 \times 2) - (.6 \times 1) = 2 + .4 - .6 = 1.8.$$

Or, written in dollars and cents, your expected winnings are $\$1.80$.

Note that the sum of the probabilities is always equal to exactly one (because all of the possible outcomes have been considered). Note also that the last term in the sum above was subtracted rather than added, because the loss of a dollar was counted as a winning of -1 dollar.

Such a game may be considered *unfair* if the expected winnings are less than the amount it costs to play. (For example, you would not want to play the game above if it cost you $\$2$ to play since you would only expect to win $\$1.80$...but for $\$1$ a game you might want to try!) Of course, most gambling games are *not* fair in this sense, which is how the owners of casinos and the governments running lotteries make a profit from the games. It is also why very few mathematicians play such games!

Let's see if you understand this concept and can apply it to the following games.

The Questions

1. ***Pick Six:*** Fifty balls are placed in a hopper. Each ball has a different one of the numbers from 1 to 50 printed on it. You have circled six of those numbers on a printed card. Then, six balls are selected from the hopper at random. If the numbers on those balls do not exactly match the circled numbers on your card, you lose the dollar you spent to play the game. If they do match, you win $\$999,999$ (which is one million dollars minus the one dollar you spent). What is your *expected winnings*? (Give your answer rounded to the nearest penny. Note that the order in which the balls appear is not relevant, only the collection of six numbers that they produce.)

   Write your answer in the box: $-\$0.94$

2. ***Single die matching:*** Roll a six-sided fair die once. If you get a "1" or "6" you win immediately. If you don't roll a "1" or "6", the number rolled is called your "spot". You roll one more time, and if you roll your spot again you win — otherwise you lose. For example, if your first roll was a "4" and your second roll was a "2", you lose. If you win, you win $\$10$. If you lose, you lose $\$8$. What is your expected winnings? (Give your answer rounded to the nearest penny.)

   Write your answer in the box: $\$0.00$

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3. ***Truncated St. Petersburg:*** Toss a fair coin repeatedly up to 10 times.
   - If you get a "Head (H)" on the first toss you win $\$2$.
   - If you get your first "H" on the second toss you win $\$4$.
   - If you get a first "H" on the third toss you win $\$8$.

     $\vdots$
   - If you get your first "H" on the tenth toss you win $\$1024$.
   - **And** if you do not get a "H" in your first 10 tosses you lose $\$10{,}500$.

   What is your expected winnings to the nearest penny?

   Write your answer in the box: $\boxed{-\$0.25}$

4. ***Queen before Spade:*** Take all the royal cards out of a 52-card deck. This gives you 12 cards: for each of the four suits (spades, clubs, diamonds and hearts) you have one of each of the three ranks of royalty (jack, queen and king). Shuffle these 12 cards and flip them over one at a time. If you get a queen before you get a spade, you win $\$10$. If you get a spade before a queen, you lose $\$3$. If you get the queen of spades before any other queen or spade you lose $\$18$. What is your expected winnings, rounded to the nearest penny?

   Write your answer in the box: $\boxed{\$1.00}$

5. ***Lottery sum:*** In this game, a deck of four cards showing the numbers 1, 2, 3 and 4 is used. A card is drawn randomly from the deck and the number on it is recorded. Then the card is replaced in the deck and the cards are shuffled. This is repeated two more times so that three numbers have been selected. Before any cards were drawn, you had to pick a number x. If the *sum* of the three numbers drawn is equal to the number x that you selected, then you win $\$x$, otherwise you lose a fixed amount of money (which does not depend on x or the cards selected.) What should you pick for x to maximize your expected winnings?

   Write your answer in the box: $\boxed{8}$