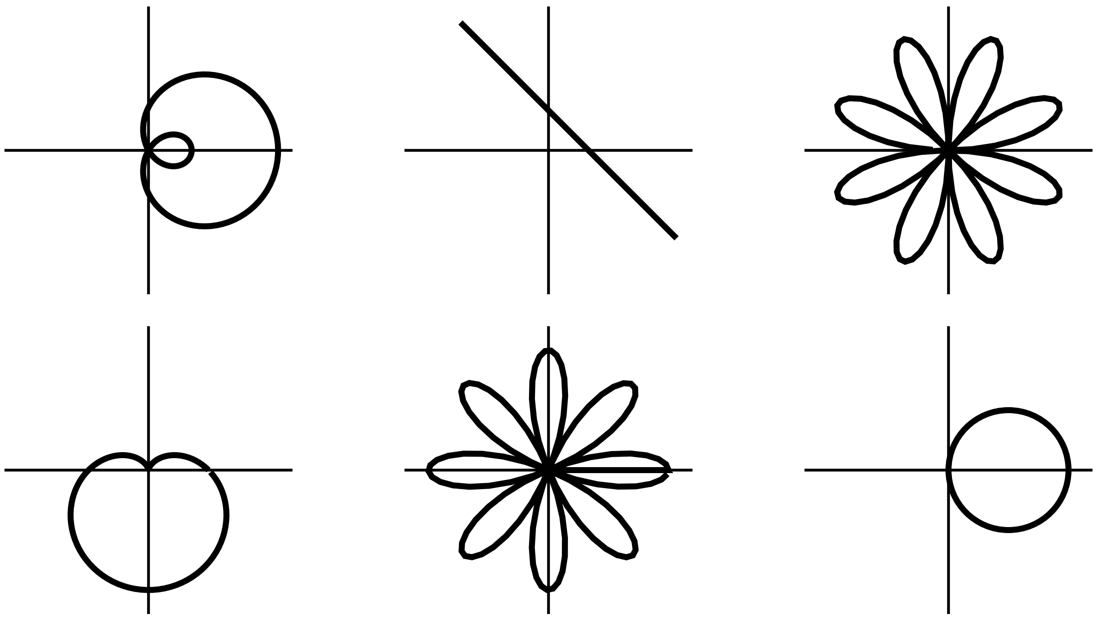
College of Charleston
Math Meet 2007
Written Test – Level 3

1. A basketball player on a local team averages 32 pounds of weight per foot of height. The weight of the player is 6 times as many pounds his coach's age in years. The sum of the player's age and the coach's age is 62 years. The coach is twice as old as the player was when the coach was half as old as the player will be when the coach is 80. How tall is the player?

   (A) 6 and $\frac{1}{2}$ feet. (B) 6 and $\frac{3}{4}$ feet. (C) 6 and $\frac{7}{8}$ feet.

   (D) 7 and $\frac{1}{8}$ feet. (E) 7 and $\frac{7}{16}$ feet.

2. Which of the following polar equations is *not* graphed in the figure?



   (A) $r = \sin(4\theta)$ (B) $r = \cos(8\theta)$

   (C) $r = 1 - \sin\theta$ (D) $r = 1 + 2\cos\theta$

   (E) $r = \dfrac{1}{\cos\theta + \sin\theta}$

3. For every positive integer n, let S_n denote the number of sequences of n 0s and/or 1s which do not contain two consecutive 1s. Suppose S_n is odd and S_{n+2} is even. Then

   (A) Both S_{n+1} and S_{n+3} must be odd.

   (B) Both S_{n+1} and S_{n+3} must be even.

   (C) S_{n+1} must be odd and S_{n+3} must be even.

   (D) S_{n+1} must be odd, but S_{n+3} could be even or odd.

   (E) S_{n+1} must be even, but S_{n+3} could be even or odd.

4. Figure 1 shows a map of eight cities, A,B,C,D,E,F,G,H, and routes for ten possible roads between the cities. A contractor wants to build as few of these roads as necessary so that it will be possible to go from any of the eight cities to any other using the finished roads. See, for example, Figure 2. How many ways are there to do this?

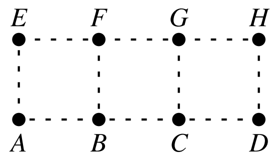


Fig. 1

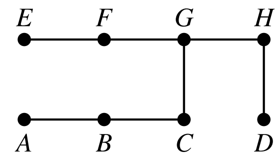


Fig. 2

(A) 120 (B) 69 (C) 56 (D) 48 (E) 35

5. $x = 0$, $y = 0$, $z = 18$ is one integer solution to the $xyz - 3xy - 2xz - yz + 6x + 2z + 3y = 36$. How many integer solutions are there altogether?

(A) 27 (B) 36 (C) 94 (D) 96 (E) 108

6. The Yankees and the Red Sox are having a playoff series to determine the American League champion. The series continues until one team has won three games, and so may take up to five games. Assuming each team has an equal chance to win each game, what is the expected number of games in the series?

(A) $\dfrac{17}{4}$ (B) $\dfrac{15}{4}$ (C) $\dfrac{25}{8}$ (D) $\dfrac{33}{8}$ (E) 4

7. A right circular cylinder of height h and radius r is continuously deformed in such a way that its volume remains constant. Which of the following is true?

(A) $\dfrac{dr}{dh}$ and $\dfrac{d^2r}{dh^2}$ are always positive.

(B) $\dfrac{dr}{dh}$ and $\dfrac{d^2r}{dh^2}$ are always negative.

(C) $\dfrac{dr}{dh}$ is always positive and $\dfrac{d^2r}{dh^2}$ is always negative.

(D) $\dfrac{dr}{dh}$ is always negative and $\dfrac{d^2r}{dh^2}$ is always positive.

(E) $\dfrac{dr}{dh}$ is always positive but $\dfrac{d^2r}{dh^2}$ could be positive or negative.

8. A teacher tried to divide a bag of pennies among her 5 favorite students, but found that she would need another two pennies to give each the same number. She then tried to divide the pennies equally among her 6 favorite students, but again she had two pennies too few, then among her 7 favorites, but again she had two pennies too few, then among her 8 favorites, but again she had two pennies too few. What's the smallest number of pennies she could have had in her bag? The hundreds digit in the correct answer is

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

9. Find the number of positive integral divisors of 104,060,401. (that is, one hundred and four million, sixty thousand, four hundred and one.)

(A) 2 (B) 5 (C) 15 (D) 27 (E) 64

10. Suppose that a, b, and c are positive integers. Find the number of points x in the interval $[0, c\pi]$ for which $\sin ax \sin bx = \cos ax \cos bx$.

(A) $a + b + 1$ (B) $2c$ (C) $ab + bc$

(D) $ca + cb$ (E) $a + b + c + 1$

11. Find the sum: $\cos 1 + \cos 2 + \cdots + \cos 101$. (All angles are measured in radians.)

(A) $\dfrac{1}{2}\left(\dfrac{\sin 101.5}{\sin 0.5} - 1\right)$ (B) $\dfrac{1}{2}\left(\dfrac{\sin 101.5}{\sin 0.5} + 1\right)$

(C) $\dfrac{1}{2}\left(\dfrac{\cos 101.5}{\cos 0.5} - 1\right)$ (D) $\dfrac{1}{2}\left(\dfrac{\sin 101.5}{\cos 0.5} + 1\right)$

(E) $\dfrac{1}{2}\left(\dfrac{\cos 101.5}{\sin 0.5} - 1\right)$

12. How many distinct real numbers x have the property that the median of the seven numbers $1.4, 1.6, 2.1, 2.2, 2.3, 2.4, x$ is equal to their mean?

(A) 0 (B) 1

(C) 2 (D) 3

(E) The are infinitely many such x.

13. When we measure the temperature at the 12 points in Fig. 1, we find that the temperature at each interior point A, B, C, or D is the average of the temperatures at its four nearest neighbors. See, for example, the tempatures in Fig. 2. What is the temperature at point A in Fig. 3?

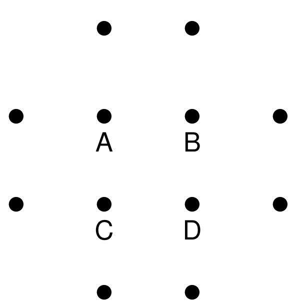


Fig. 1

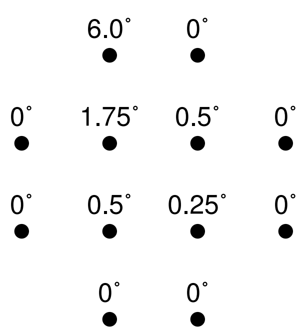


Fig. 2

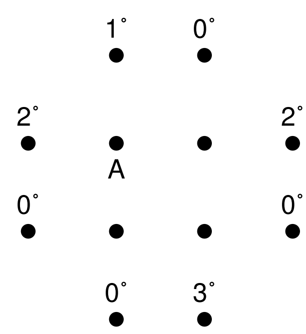


Fig. 3

(A) $5/12^\circ$ (B) $7/6^\circ$ (C) $5/6^\circ$ (D) $13/12^\circ$ (E) $7/12^\circ$

14. Find $\dfrac{dy}{dx}$ if $y = \ln(\sqrt{x^2+9} - 3) - 2\ln\sqrt{x}$.

(A) $\dfrac{-3}{x\sqrt{x^2+9}}$

(B) $\dfrac{-x}{\sqrt{x^2+9}(\sqrt{x^2+9}+3)} + \dfrac{1}{x}$

(C) $\dfrac{-9}{x(x^2+9)}$

(D) $\dfrac{3}{\sqrt{x}(x^2+9)}$

(E) $\dfrac{3 - x + \sqrt{x^2+9}}{x^2}$

15. Points A, B, C lie on the circumference of a circle of radius 1. If the length of arc ABC is 2, find the radian measure of the angle ABC (that is, the angle at B between the two line segments AB and BC).

(A) $\pi - 1$

(B) $\pi - 2$

(C) $1 + \pi/4$

(D) $2 - \pi/3$

(E) Not enough information is given to determine the angle.

16. Define the sequence $\{b_n\}$ by $b_0 = 1$, $b_1 = 2$, and $b_{n+2} = b_{n+1} + b_n$. Which of the following approximations best describes b_n? (Here C is an unspecified constant.)

(A) $b_n \approx C \cdot \left(\dfrac{\sqrt{11}}{2}\right)^n$ (B) $b_n \approx C \cdot \left(\dfrac{-1+\sqrt{7}}{2}\right)^n$

(C) $b_n \approx C \cdot \left(\dfrac{1+\sqrt{5}}{2}\right)^n$ (D) $b_n \approx C \cdot \left(\dfrac{2}{\sqrt{6}}\right)^n$

(E) $b_n \approx C \cdot \left(\dfrac{2}{1+\sqrt{3}}\right)^n$

17. If $f(x) = e^x - e^{-x}$, then $f^{-1}(x) =$

(A) $\ln(x + \sqrt{x^2+4}) - \ln 2$ (B) $\ln(x + \sqrt{x^2-4}) + \ln 2$

(C) $\ln(x - \sqrt{x^2+4}) - \ln 2$ (D) $\ln(x - \sqrt{x^2-4}) - \ln 2$

(E) $\ln(x + \sqrt{x^2+4}) + \ln 2$

18. Find the greatest slope along the graph of $4x^2 - y^2 + 2y - 1 = 0$.

(A) -3 (B) -2 (C) 2 (D) 3 (E) 4

19. A cone is formed by rotating the line $2y = x$ about the y-axis. A sphere of radius 20 is dropped in the cone. At what y-value does the sphere touch the cone?

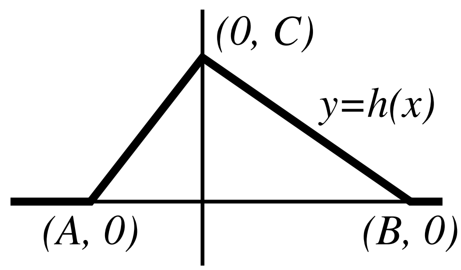
(A) $-1+\sqrt{5}$ (B) $1+\sqrt{5}$ (C) $2\sqrt{5}$

(D) $\dfrac{\sqrt{5}}{5}$ (E) $\sqrt{500}$

20. Let $h(x)$ be the piecewise-linear function graphed in the figure below. Fact: There exist numbers α, β, and γ so that

$$\int_A^B h(x)f''(x)\,dx = \alpha f(A) + \beta f(B) + \gamma f(0)$$

for any twice continuously differentiable function f. Find $\alpha+\beta+\gamma$.



(A) $(A+B)/C$ (B) C/B (C) $C(A+B)/2$

(D) $C/B - C/A$ (E) 0

21. If Paul has fish for dinner, then he'll either pay his rent or read a book (or both). If he pays his rent, then he'll sleep in on Monday. If he does not read a book, then he'll not sleep in on Monday. Paul does not sleep in on Monday. What can you conclude?

(A) Paul paid his rent.

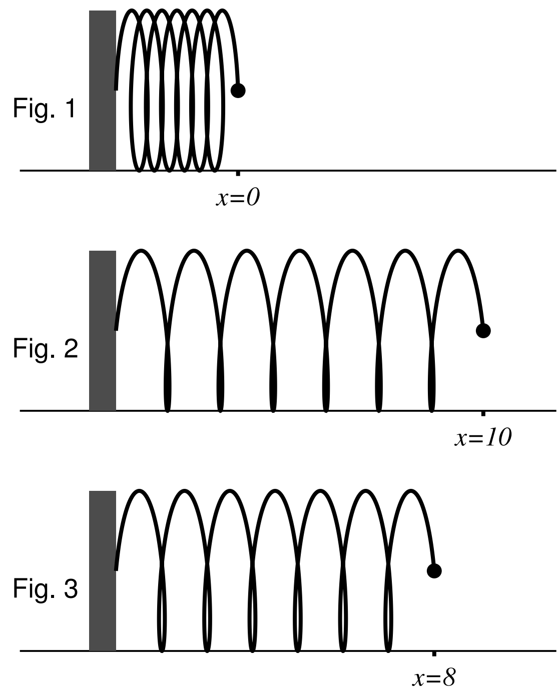
(B) If he does not have fish for dinner, then he'll read a book.

(C) If he pays his rent, then he'll *not* have fish for dinner.

(D) If he reads a book, then he'll sleep late on Monday.

(E) The sentence in answer d. is false.

22. A tightly compressed spring lies along the x-axis. When the spring is released, it expands until its free end (●) moves from position $x = 0$ (Fig. 1) to $x = 10$ (Fig. 2). The spring then contracts until the free end is at position $x = 8$ (Fig. 3). The spring then expands, then contracts, then expands, etc., by decreasingly smaller amounts; each time the free end moves to the right (or left) only 20% as far as it last moved to the left (or right). Where will the free end eventually come to rest?



(A) $x = 25/3$ (B) $x = 49/6$ (C) $x = 33/4$
(D) $x = 65/8$ (E) $x = 42/5$

23. Suppose we divide all the Math Meet participants into boy-girl pairs, and we send each pair of students to stand at a randomly chosen point on the CofC gym floor. Suppose Professor Kasman paints a blue dot on the floor at a randomly selected point, and Professor Caveny paints a white dot at another point. Then we instruct the boy in each pair to walk in a straight line to the blue dot, make a 90° left turn, walk in a straight line the same distance he just traveled, and stop. Then we instruct the *girl* in each pair to walk in a straight line to the *white* dot, make a 90° *right* turn, walk in a straight line the same distance *she* just traveled, and stop. Then we instruct each boy and girl to walk toward their partner in a straight line at identical speeds, and to paint a red dot on the floor at the point where the two meet. Which best describes the resulting collection of red dots?

(A) The red dots form a straight line.

(B) The red dots form a hyperbola.

(C) The red dots form parabola.

(D) The red dots form an ellipse.

(E) The red dots are all at the same point.

24. If the graph of $y = f(x)$ is symmetric about both the lines $x = 0$ and $x = 2$, then which of the following equals $f(x)$?

(A) $\dfrac{f(x) - f(-x)}{2}$

(B) $\dfrac{f(x+2) + f(-x+2)}{2}$

(C) $\dfrac{f(x+4) - f(4-x)}{2}$

(D) $\dfrac{f(x) + f(x-2) - f(x+2) + f(x+4)}{2}$

(E) $\dfrac{f(x) - f(x-2) + f(x+2) - f(x-4)}{2}$

25. The makers of FatBlasters$^{\mathrm{TM}}$ dietary supplements wish to design a new product out of three ingredients: Green Tea, Ginko Biloba, and Swampwort. 1 g of Green Tea contains 1 mg of vitamin A, 3 mg of vitamin B, and 7 mg of vitamin C. 1 g of Ginko Biloba contains 2 mg of A, 8 mg of B, and 16 mg of C. 1 g of Swampwort contains 5 mg of A, 18 mg of B, and 38 mg of C. The new supplement should contain exactly 6 mg of vitamin A, 21 mg of B, and 45 mg of C. How much Swampwort must go into the new product?

(A) There must be 1.166 g of swampwort.

(B) There can be any amount between 0 and 1.2 g inclusive.

(C) There can be any amount between 0 and 1.166 g inclusive.

(D) There can be any amount between 0 and 1.0 g inclusive.

(E) There can be any amount of swampwort.

2007 Answers / Level 3 Test

1. D
2. B
3. A
4. C
5. E
6. D
7. D
8. E
9. B
10. D
11. A
12. B
13. B
14. B
15. A
16. C
17. A
18. C
19. C
20. E
21. C
22. A
23. E
24. D
25. D