BRIEF ANSWERS TO MATH MARATHON 2013

1. What is the smallest possible surface area for a rectangular box with sides of integer length and volume 2013 cubic units?

The prime factorization of 2013 is $3 \times 11 \times 61$; these dimensions give the minimal surface area of $2((3 \times 11) + (3 \times 61) + (11 \times 61)) = 1774$. Other configurations of the box, such as $1 \times 33 \times 61$, give much larger surface areas.

2. The random number generator on a computer is used to generate two numbers a, b in the interval $[0, 10]$. What is the probability that

$$\log_b(\log_b(\log_b a))$$

is defined?

This expression will be defined when either $a > b > 1$ or $0 < b < a < 1$. The area of the square $0 \leq a, b \leq 10$ is 100 square units, and the sum of the areas of the two triangles $1 < b < a \leq 10$ and $0 < b < a < 1$ is $\frac{81}{2} + \frac{1}{2} = 41$ square units, so the probability is 41%.

3. I have a set of squares with integer sides whose average area is 2013. This set consists of as few squares as possible such that the above condition is satisfied; furthermore the average perimeter of these squares is as small as possible such that the above conditions are satisfied. What are the squares in this set?

This is probably the most "marathon" of this year's questions. No fewer than four squares will suffice. 4026 is not a sum of two squares (any square is 0, 1, 4, or 7 modulo 9, but $4026 \equiv 3 \pmod 9$), and 6039 is not a sum of three squares (any square is 0, 1, or 4 modulo 8, but $6039 \equiv 7 \pmod 8$). There are many expressions of 8052 as a sum of four squares; the minimum average perimeter results from maximizing the largest square. A "greedy" approach then gives the squares $89^2, 11^2, 3^2, 1^2$.

4. For some non-zero constants a, b, c, and d, the function

$$f(x) = \frac{ax + b}{cx + d}$$

is its own inverse, and the function $g(x) = f(1/x)$ is also its own inverse. If $f(2013) = 2013$, find $f(-1) + f(0) + f(1)$.

The two "inverse" conditions imply $a = -d$ and $b = -c$. This implies $f(1) = -1$ and $f(-1) = 1$. So the requested sum is just $f(0) = -b/a$. The condition $f(2013) = 2013$ gives $b/a = \frac{-2013}{2026085}$, so the requested sum is $\frac{2013}{2026085}$.

5. A game is played on the vertices of the cube $\{(x, y, z) : 0 \leq x, y, z \leq 1\}$ as follows. Beginning at $(0, 0, 0)$, the player moves along one of the edges emanating from that vertex to an adjacent vertex according to a roll of a die; if a 1 or 2 is rolled, she moves in the x-direction; if a 3 or 4 is rolled, in the y-direction; and if a 5 or 6 is rolled, in the z-direction. The player continues to move in this manner, according to the roll of a die, until

she reaches either (1,1,0) or (1,1,1). If she reaches (1,1,1), the game is won; if she reaches (1,1,0), the game is lost. What is the probability that she wins the game?

Let p be the requested probability. let x denote the probability of winning if she is at $(1, 0, 1)$; by symmetry, x is also the probability from $(0, 1, 1)$. Let y be the probability of winning from either $(0, 1, 0)$ or $(1, 0, 0)$ (the same by symmetry), and let z be the probability from $(1, 0, 0)$. Since the probability of moving from a vertex to any adjacent vertex is always $\frac{1}{3}$, we get the system of equations

$$p = \frac{2}{3}y + \frac{1}{3}z;$$

$$x = \frac{1}{3}y + \frac{1}{3}z + \frac{1}{3};$$

$$y = \frac{1}{3}p + \frac{1}{3}x;$$

$$z = \frac{1}{3}p + \frac{1}{3}x + \frac{1}{3}y.$$

The solution to this system is $(p, x, y, z) = (\frac{3}{7}, \frac{9}{14}, \frac{5}{14}, \frac{4}{7})$; the requested probability is 3/7.