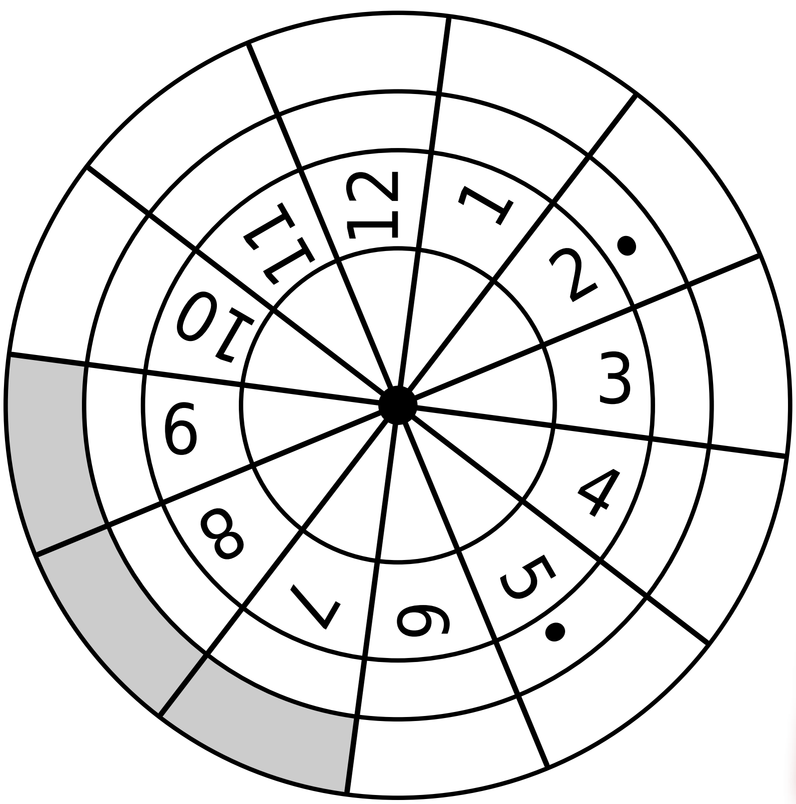
Clock of Doom All-Day Sprint

CofC Math Meet 2014
THE NUMBER GAMES

The $3 \cdot 5^2$-th Number Games had begun. After running through a forest of binary trees, tripping over square roots, and dodging convergent nets, Katenary Evenmean and her friend Pitau arrived at the abandoned cone-u-copia at the center of the field and took a moment to catch their breaths.

“Look,” said Pitau, “there’s a device hidden under the cone-u-copia. That must the control panel we were told about. Maybe we can break into it and open the spherical boundary.”

Katenary nodded, and used a wedge to pry it open. “It looks like a clock,” she said. “There are instructions. . . ”



Instructions for Opening Clock of Doom Access Panel

- Enter a + or · in each sector of the second ring. There are four + signs and eight · signs. The two · signs in sectors 2 and 5 are fixed and cannot be changed.
- Enter the numbers 1 through 12 in the sectors of the outer ring.
- Each number is used exactly once.
- The product of the two numbers in a sector with a · must be a multiple of 12. The sum of the two numbers in a sector with a + must be a multiple of 12. The three numbers paired with 7, 8, and 9 (shaded area) sum to exactly 12.

After tinkering for a while, Katenary and Pitau managed to set the clock just right, and sure enough, the side of the device opened, revealing another mechanism and more instructions:

Clock of Doom Boundary Control

To open the boundary of the arena sphere, enter in the space below all pairs of integers (a_1, a_2) such that

- $1 \leq a_1 \leq 11$ and $1 \leq a_2 \leq 11$
- the polynomial $f(x) = a_1 x + a_2 x^2$, when interpreted as a function on the integers modulo 12, is a bijection.

"Bijection? Modulo 12?" asked Pitau, "What does all that mean?"

Katenary replied, "The modulo 12 part means that you do integer arithmetic, but as if you're working on a clock, so you reduce every result to its remainder when divided by 12. For instance, 8+8 $\equiv$ 16 $\equiv$ 4 (mod 12), and 9·4 $\equiv$ 36 $\equiv$ 0 (mod 12). The bijection part means that f is a one-to-one correspondence. In other words, the output value of f cannot be equal for two different inputs between 0 and 11."

"This sounds hard," Pitau said. "I can see that $f(0) \equiv 0$ no matter what a_1 and a_2 are, so that's a start. But is there a way to find these polynomials without checking every possible one? How many are we supposed to find?"

"I don't know," she replied. "There's space to enter several. . . "