Egyptian Fractions

The ancient Egyptians are notorious for their peculiar system of fractions. They insisted on using *unit fractions*, which have 1 in the numerator. Any fractional amount had to be expressed as a sum of *distinct* unit fractions. So instead of $^2/_5$ or $^1/_5 + {}^1/_5$, they would have written

$$\frac{1}{3}+\frac{1}{15}$$

There were a few exceptions: They had special symbols for $^2/_3$ and $^3/_4$. As you can imagine, their arithmetic could get really complicated. The Rhind papyrus, which is the source of the image on this year's tee shirt and poster, includes tables of how to express numbers of the form $^2/_n$ as sums of distinct unit fractions.

Every rational number between 0 and 1 can be expressed in this form in at least one way. Here's one way to prove this. Leonardo of Pisa, commonly known as Fibonacci and famous for the sequence $1,1,2,3,5,8\ldots$, included a section in his book *Liber Abaci* on Egyptian fractions. It included the following procedure for expanding a fraction of the form $^m/_n$ into a sum of distinct unit fractions.

- Let $k = \lceil ^n/_m \rceil$ that is, the least integer greater than or equal to $^n/_m$. The point of this is that $^1/_k$ is the largest unit fraction such that $^1/_k \le {}^m/_n$.
- Let the remainder be $^{m'}/_{n'} = {}^m/_n - {}^1/_k$. Now

$$\frac{m}{n}=\frac{1}{k}+\frac{m'}{n'}$$

- Apply the process to $^{m'}/_{n'}$ and repeat until the remainder has numerator 1.
- The sum of $^1/_k$ over all the k's is the expression for the original $^m/_n$.

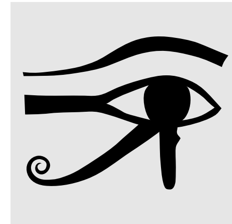
Later, it was proved that after each iteration, the numerator of the remainder must decrease, that is, it is always true that $m' < m$, so the process has to stop eventually. This algorithm sometimes results in expressions that are far more complicated than necessary, with lots of terms and huge denominators, but it always works.

Here's an example of the procedure in action on $^4/_{13}$

$$\left\lceil\frac{13}{4}\right\rceil=\left\lceil 3+\frac{1}{4}\right\rceil=4$$
$$\frac{4}{13}-\frac{1}{4}=\frac{3}{52}$$
$$\frac{4}{13}=\frac{1}{4}+\frac{3}{52}$$

$$\left\lceil\frac{52}{3}\right\rceil=\left\lceil 17+\frac{1}{3}\right\rceil=18$$
$$\frac{3}{52}-\frac{1}{18}=\frac{1}{468}$$
$$\frac{3}{52}=\frac{1}{18}+\frac{1}{468}$$

$$\frac{4}{13}=\frac{1}{4}+\frac{1}{18}+\frac{1}{468}$$



Here are some problems involving Egyptian fractions.

1. Express $5/11$ as a sum of distinct unit fractions:

$$\frac{5}{11} =$$

2. Express $7/12$ as a sum of distinct unit fractions in two different ways:

$$\frac{7}{12} = \qquad \frac{7}{12} =$$

3. Find positive integers A and B such that the polynomial

$$f(x) = Ax^2 - Bx + 1$$

has two distinct roots r_1 and r_2 that are positive rational numbers, and their sum is $r_1 + r_2 = 5/19$.

$A =$ $B =$

4. Find positive integers A, B, and C such that the polynomial

$$f(x) = Ax^3 - Bx^2 + Cx - 1$$

has three distinct roots that are positive rational numbers, and their sum is $7/22$. There are at least 5 such triples (A, B, C). How many can you find?

$A =$ $B =$ $C =$

$A =$ $B =$ $C =$

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$A =$ $B =$ $C =$

$A =$ $B =$ $C =$