COLLEGE of CHARLESTON

MATH MEET 2017

Math Marathon

Instructions

- The problems are to be worked out individually and independently. Only textbooks and library sources may be used. Calculators and computers may be used. Each entry must be signed by a math teacher within the school to certify that all rules have been followed. Any number of entries from a school may be submitted.
- Work must be shown neatly and concisely. Explain how you got your answer. It is possible that several entries will have correct solutions, so work will be judged on exposition, clarity of thought and ingenuity, as well as correctness. The date of submission will also be considered. Electronic submissions will be accepted only once.
- All entrants must be students who have not graduated from high school. All entrants must be registered for the Math Meet.
- The judges' decisions will be final.
- All papers are to be submitted electronically to `mathmeet@cofc.edu` or mailed to the following address

  Math Meet (Marathon)
  Department of Mathematics
  College of Charleston
  66 George Street
  Charleston, SC 29424

- The cover paper for each entry must have the following information: (This may be turned in the day of the Math Meet if submitted electronically and not mailed.) Student Name, Math Marathon, Home Address, E-mail Address, School; Year of Graduation, School Address, Signature of a Math Teacher for Verification.
- All entries must be received or postmarked by February 1, 2017.

Marathon Problems

1. Find all solutions to the equation $a^b + c^d = 2017$ in distinct integers a, b, c, d all greater than 1.

2. The random number generator on a computer is used to generate three numbers a, b, c uniformly in the interval $[0, x]$, where $x \geq 1$. Let Y denote the number of the three values

$$\log_a(\log_b c), \qquad \log_b(\log_c a), \qquad \log_c(\log_a b)$$

which are defined. Express the expected value of Y as a function of x.

3. Find all solutions to the equation $2017 = x^m - y^m$ where x, y, m are positive integers and $m > 1$.

4. Suppose that $\cos\theta = 2\cos 2\theta$. Find rational numbers A and B so that

$$\tan^4\theta = A\tan^2\theta + B.$$

5. The graph of the hyperbola

$$y = \frac{1}{2x}$$

is tangent to the unit circle $x^2 + y^2 = 1$ at two distinct points, and the distance between these two points is exactly 2 units. If I choose nonzero constants a, b, c, d so that the graph of the rational function

$$y = \frac{ax + b}{cx + d}$$

is tangent to the unit circle at two distinct points, express the distance between these two points as a function of a, b, c, d.

Answers in brief

1. The solutions are $2017 = 3^4 + 44^2$ and $2017 = 17^2 + 12^3$.

2. The expected value is

$$3 - \frac{6}{x} + \frac{6}{x^2}.$$

3. The only solution is $x = 1009,\ y = 1008,\ m = 2$.

4. The constants are

$$A = \frac{9}{4}, \qquad B = -\frac{3}{4}.$$

5. The distance is

$$d = \sqrt{\frac{2a^2}{c^2} + \frac{8\sqrt{b^2 - a^2}}{|c|}}.$$