COLLEGE of CHARLESTON
1770
MATH MEET 2018

Snowflake Symmetries: Background

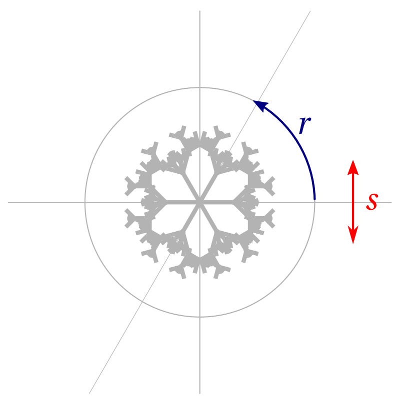
This sheet is the background information you'll need for the snowflake symmetries all-day sprint.

Imagine all the things you can to do the (x, y) plane that leave the origin in place: You can rotate by any angle about the origin, either direction. You can flip the plane over. You can stretch it and shrink it.

But if you center a snowflake at the origin, there are only twelve distinct operations you can do to the plane that preserve the snowflake. These are its symmetries.

You can leave everything in place, or turn the plane by $1/6$ of a turn, or $2/6$, $\ldots 5/6$. You can also flip the plane through each of 6 different lines of mirror symmetry, three through opposite vertices and three through opposite sides. You can build all of the symmetries from two basic operations:

- r means to rotate by $\pi/3$ radians (60 degrees) counterclockwise
- s means to reflect across the horizontal axis.



To rotate by other angles, apply r multiple times. It's helpful to shorten repeated applications of r by using a superscript. Then, it's even sensible to use zero or negative superscripts. So, for example:

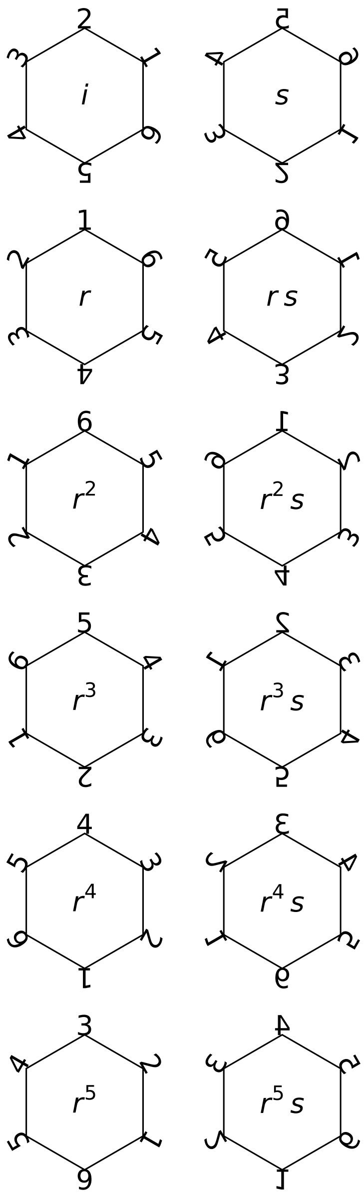
$$r(\ding{43}) = \rotatebox{60}{\ding{43}}$$

$$r^2(\ding{43}) = r(r(\ding{43})) = \rotatebox{120}{\ding{43}}$$

$$r^3(\ding{43}) = r(r(r(\ding{43}))) = \rotatebox{180}{\ding{43}}$$

$$r^0(\ding{43}) = \ding{43}$$

$$r^{-1}(\ding{43}) = \rotatebox{-60}{\ding{43}}$$



It's easiest to see how these operations work if we draw a plain hexagon and number the vertices. We can write all of the operations that preserve the hexagon or snowflake as compositions of r and s. Let's use i for the identity operation, that is, leaving the plane unchanged. We may as well use a little shorthand, inspired by the r^n notation, and drop parentheses when possible:

$$r^2 s(\ldots) = r(r(s(\ldots)))$$

The twelve operations, that is, the twelve symmetries of a snowflake, are illustrated to the left.

It's possible to consider other compositions of r and s, for example, rsr^2. But notice that $sr = r^5 s$, so we can always push the s all the way to the right. Also, $r^6 = r^0 = i$ because rotating the plane by $6 \times 60 = 360$ degrees is the same as leaving it in place. Likewise, $s^2 = i$, because reflecting the plane across the horizontal axis and then reflecting it again puts it back in place. That means we never need more than one s or more than five r's.

Back to the example:

$$rsr^2 = rsrr = rr^5 sr = rr^5 r^5 s = r^6 r^5 s = ir^5 s = r^5 s$$