Euclid's algorithm is a way to find the greatest common divisor (GCD) of two integers. Here's the core idea. Given \( a \) and \( b \), with \( a > b \), use division with remainder to write

\[
\begin{align*}
a &= bq + r, \quad 0 \leq r < b \tag{1}
\end{align*}
\]

or equivalently

\[
\begin{align*}
a - bq &= r \tag{2}
\end{align*}
\]

From (2), you can see that the GCD of \( a \) and \( b \) must also divide \( r \). From (1), you can see that the GCD of \( b \) and \( r \) must also divide \( a \). Putting all that together, the GCD of \( a \) and \( b \) must also be the GCD of \( b \) and \( r \). Since \( r \) is less than \( b \), we've narrowed down the possibilities for the GCD considerably. Now all you have to do is divide \( b \) by \( r \), and keep repeating this process until you hit a remainder of zero. The last non-zero remainder is the GCD you're looking for.

Here's an example. Let's find the GCD of 288 and 120. First,

\[
\begin{align*}
288 &= 120 \times 2 + 48
\end{align*}
\]

So \( \text{gcd}(288, 120) = \text{gcd}(120, 48) \). Repeat:

\[
\begin{align*}
120 &= 48 \times 2 + 24 \\
48 &= 24 \times 2
\end{align*}
\]

So \( \text{gcd}(288, 120) = \text{gcd}(120, 48) = \text{gcd}(48, 24) = 24 \). And sure enough, 288 = 24 \times 12 and 120 = 24 \times 5.

1. Use Euclid’s algorithm to find \( \text{gcd}(720, 520) \).

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   \]

   Interestingly, the steps of Euclid’s algorithm allow you to find integers \( m \) and \( n \) that solve

\[
\begin{align*}
am + bn &= \text{gcd}(a, b) \tag{3}
\end{align*}
\]

Here's how it works for \( a = 288 \) and \( b = 120 \). Go back through the calculations and isolate the remainder in each step:

\[
\begin{align*}
288 - 120 \times 2 &= 48 \\
120 - 48 \times 2 &= 24 \tag{4}
\end{align*}
\]
Use the first step to replace the 48 in the second step

\[ 120 - (288 - 120 \times 2) \times 2 = 24 \]

and simplify to get

\[ 120 \times 5 + 288 \times (-2) = 24 \]

Here's another example. Let's compute gcd(200, 29) = 1.

\[
\begin{align*}
200 &= 29 \times 6 + 26 \\
29 &= 26 \times 1 + 3 \\
26 &= 3 \times 8 + 2 \\
3 &= 2 \times 1 + 1 \\
2 &= 1 \times 2
\end{align*}
\]

Now we go bottom up:

\[
\begin{align*}
3 - 2 \times 1 &= 1 \\
3 - (26 - 3 \times 8) \times 1 &= 26 \times (-1) + 3 \times 9 = 1 \\
26 \times (-1) + (29 - 26 \times 1) \times 9 &= 29 \times 9 + 26 \times (-10) = 1 \\
29 \times 9 + (200 - 29 \times 6) \times (-10) &= 200 \times (-10) + 29 \times 69 = 1
\end{align*}
\]

So a solution in integers of 200m + 29n = 1 is m = -10, n = 69.

2. Find integers \(m\) and \(n\) such that \(720m + 520n = \text{gcd}(720, 520)\).

\[
m = -5, \ n = 7
\]

3. Find integers \(m\) and \(n\) such that \(320m + 119n = 1\).

\[
m = 45, \ n = -121
\]

4. Find an integer \(0 \leq k < 320\) such that \(119k \equiv 1 \pmod{320}\).

\[
k = 199
\]