



Modular Arithmetic All-Day Sprint

Modular arithmetic is an alternative way of doing operations on integers. The idea is that you pick a modulus $n > 1$, and say that two numbers are congruent modulo n if they differ by an integer multiple of n . So for example, all odd numbers are congruent to 1 modulo 2. The numbers 2, 12, -8 , and 65892 are all congruent modulo 10. The usual notation is:

$$2 \equiv 12 \pmod{10}$$

Every integer is congruent to exactly one integer that is both at least 0 and less than n , so it's typical to think of that number as the standard representative of all the numbers it's congruent to. Common notation for that is to use "mod" as an infix symbol:

$$x \bmod n = \text{the integer } j \text{ such that } 0 \leq j < n \text{ and } j \equiv x \pmod{n}$$

It's easy to find j . Just divide x by n and keep the remainder. So for example:

$$13 \equiv 1 \pmod{6} \quad \text{because } 13 = 6 \times 2 + 1 \text{ and } 13 - 1 = 6 \times 2$$

It's easy to prove (and I encourage you to try) that if $a \equiv x \pmod{n}$ and $b \equiv y \pmod{n}$, then $a + b \equiv x + y \pmod{n}$ and $ab \equiv xy \pmod{n}$.

There's a helpful trick for computing powers in modular arithmetic. Suppose you need $10^{500} \bmod 17$. You can compute

$$10^2 = 100 \equiv 15 \pmod{17}$$

If you square again,

$$10^4 \equiv 15^2 \equiv 225 \equiv 4 \pmod{17}$$

Keep squaring:

$$10^8 \equiv 4^2 \equiv 16 \pmod{17}$$

$$10^{16} \equiv 16^2 \equiv 1 \pmod{17}$$

Continuing the process of squaring, it's clear that $10^{32} \equiv 10^{64} \equiv \dots \equiv 1 \pmod{17}$. Since $500 = 16 \times 31 + 4$,

$$10^{500} \equiv (10^{16})^{31} \times 10^4 \equiv 4 \pmod{17}.$$

Similarly, since $6887 = 16 \times 430 + 4 + 2 + 1$,

$$10^{6887} \equiv (10^{16})^{430} \times 10^4 \times 10^2 \times 10^1 \equiv 1^{430} \times 4 \times 15 \times 10 \equiv 5 \pmod{17}$$

1. Find the standard representative of $30 + 13 \times 899 - 55^8 + 903796 \pmod{2}$

0

2. Find the standard representative of $30 + 13 \times 899 - 55^8 + 903796 \pmod{9}$

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3. Find the standard representative of $30 + 13 \times 899 - 55^8 + 903796 \pmod{7}$

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4. Find the standard representative of $3^{22707} \pmod{31}$. Hint: If you can find a $k > 0$ such that $3^k \equiv 1 \pmod{31}$ that will save you a lot of work.

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