Modular Arithmetic All-Day Sprint

Modular arithmetic is an alternative way of doing operations on integers. The idea is that you pick a modulus \( n > 1 \), and say that two numbers are congruent modulo \( n \) if they differ by an integer multiple of \( n \). So for example, all odd numbers are congruent to 1 modulo 2. The numbers 2, 12, −8, and 65892 are all congruent modulo 10. The usual notation is:

\[ 2 \equiv 12 \pmod{10} \]

Every integer is congruent to exactly one integer that is both at least 0 and less than \( n \), so it's typical to think of that number as the standard representative of all the numbers it's congruent to. Common notation for that is to use “mod” as an infix symbol:

\[ x \mod n = \text{the integer } j \text{ such that } 0 \leq j < n \text{ and } j \equiv x \pmod{n} \]

It's easy to find \( j \). Just divide \( x \) by \( n \) and keep the remainder. So for example:

\[ 13 \equiv 1 \pmod{6} \quad \text{because } 13 = 6 \times 2 + 1 \text{ and } 13 - 1 = 6 \times 2 \]

It's easy to prove (and I encourage you to try) that if \( a \equiv x \pmod{n} \) and \( b \equiv y \pmod{n} \), then \( a + b \equiv x + y \pmod{n} \) and \( a b \equiv x y \pmod{n} \).

There's a helpful trick for computing powers in modular arithmetic. Suppose you need \( 10^{500} \pmod{17} \). You can compute

\[ 10^2 = 100 \equiv 15 \pmod{17} \]

If you square again,

\[ 10^4 \equiv 15^2 \equiv 225 \equiv 4 \pmod{17} \]

Keep squaring:

\[ 10^8 \equiv 4^2 \equiv 16 \pmod{17} \]
\[ 10^{16} \equiv 16^2 \equiv 1 \pmod{17} \]

Continuing the process of squaring, it's clear that \( 10^{32} \equiv 10^{64} \equiv \ldots \equiv 1 \pmod{17} \). Since \( 500 = 16 \times 31 + 4 \),

\[ 10^{500} \equiv (10^{16})^{31} \times 10^4 \equiv 4 \pmod{17} \]

Similarly, since \( 6887 = 16 \times 430 + 4 + 2 + 1 \),

\[ 10^{6887} \equiv (10^{16})^{430} \times 10^4 \times 10^2 \times 10^1 \equiv 1^{430} \times 4 \times 15 \times 10 \equiv 5 \pmod{17} \]
1. Find the standard representative of $30 + 13 \times 899 - 55^8 + 903796 \mod 2$

2. Find the standard representative of $30 + 13 \times 899 - 55^8 + 903796 \mod 9$

3. Find the standard representative of $30 + 13 \times 899 - 55^8 + 903796 \mod 7$

4. Find the standard representative of $3^{22707} \mod 31$. Hint: If you can find a $k > 0$ such that $3^k \equiv 1 \pmod{31}$ that will save you a lot of work.