COLLEGE of CHARLESTON
1770
MATH MEET 2019

Modular Arithmetic All-Day Sprint

Modular arithmetic is an alternative way of doing operations on integers. The idea is that you pick a modulus $n > 1$, and say that two numbers are congruent modulo n if they differ by an integer multiple of n. So for example, all odd numbers are congruent to 1 modulo 2. The numbers 2, 12, −8, and 65892 are all congruent modulo 10. The usual notation is:

$$2 \equiv 12 \pmod{10}$$

Every integer is congruent to exactly one integer that is both at least 0 and less than n, so it's typical to think of that number as the standard representative of all the numbers it's congruent to. Common notation for that is to use "mod" as an infix symbol:

$$x \bmod n = \text{ the integer } j \text{ such that } 0 \le j < n \text{ and } j \equiv x \pmod{n}$$

It's easy to find j. Just divide x by n and keep the remainder. So for example:

$$13 \equiv 1 \pmod{6} \quad \text{because } 13 = 6 \times 2 + 1 \text{ and } 13 - 1 = 6 \times 2$$

It's easy to prove (and I encourage you to try) that if $a \equiv x \pmod{n}$ and $b \equiv y \pmod{n}$, then $a + b \equiv x + y \pmod{n}$ and $ab = xy \pmod{n}$.

There's a helpful trick for computing powers in modular arithmetic. Suppose you need 10^{500} mod 17. You can compute

$$10^2 = 100 \equiv 15 \pmod{17}$$

If you square again,

$$10^4 \equiv 15^2 \equiv 225 \equiv 4 \pmod{17}$$

Keep squaring:

$$10^8 \equiv 4^2 \equiv 16 \pmod{17}$$
$$10^{16} \equiv 16^2 \equiv 1 \pmod{17}$$

Continuing the process of squaring, it's clear that $10^{32} \equiv 10^{64} \equiv \ldots \equiv 1 \pmod{17}$. Since $500 = 16 \times 31 + 4$,

$$10^{500} \equiv \left(10^{16}\right)^{31} \times 10^4 \equiv 4 \pmod{17}.$$

Similarly, since $6887 = 16 \times 430 + 4 + 2 + 1$,

$$10^{6887} \equiv \left(10^{16}\right)^{430} \times 10^4 \times 10^2 \times 10^1 \equiv 1^{430} \times 4 \times 15 \times 10 \equiv 5 \pmod{17}$$

1. Find the standard representative of $30+13\times 899-55^8+903796 \bmod 2$

2. Find the standard representative of $30+13\times 899-55^8+903796 \bmod 9$

3. Find the standard representative of $30+13\times 899-55^8+903796 \bmod 7$

4. Find the standard representative of $3^{22707} \bmod 31$. Hint: If you can find a $k>0$ such that $3^k \equiv 1 \pmod{31}$ that will save you a lot of work.