COLLEGE of CHARLESTON

1770

MATH MEET 2020

Cantor Set 2020 Sprint

The standard Cantor set C, named after mathematician George Cantor, is defined as follows. Start with the interval $C_0 = [0,1]$. Cut out the central third, which leaves two intervals, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Now cut out the central third of each of those intervals, which leaves four intervals, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. And keep going forever. The points that are left form the Cantor set $C = \bigcap_k C_k$.

The total length of C is zero. To see this, note that at step k, C_k consists of 2^k intervals of length $\left(\frac{1}{3}\right)^k$. So the total length of C_k must be $\left(\frac{2}{3}\right)^k$. But that shrinks to zero as k increases. The total length of C can be no more than that of any of the sets C_k, so it must be zero!

There's a straightforward way to tell if a number x is in C, and it has to do with how x can be expressed in base 3 as a series of digits. Writing a number $x \in [0,1]$ as a base 3 series of digits means you express x in the form

$$x = \frac{d_1}{3} + \frac{d_2}{3^2} + \frac{d_3}{3^3} + \cdots = \langle 0.d_1 d_2 d_3 \ldots \rangle_3$$

where each $d_j \in \{0,1,2\}$. For example,

$$\frac{1}{3} = \langle 0.1 \rangle_3 \quad \frac{2}{3} = \langle 0.2 \rangle_3 \quad \frac{1}{9} = \langle 0.01 \rangle_3 \quad \frac{2}{9} = \langle 0.02 \rangle_3 \quad \frac{5}{9} = \langle 0.12 \rangle_3$$

Here's how to convert a number x to its base 3 digits. Multiplying $x = \langle 0.d_1 d_2 d_3 \ldots \rangle_3$ by 3 yields $3x = \langle d_1.d_2 d_3 \ldots \rangle_3$ which means that the integer part of $3x$ is the first digit, $d_1 = \lfloor 3x \rfloor$, and the fractional part is $3x - d_1 = \left\langle 0.d_2 d_3 d_4 \ldots \right\rangle_3$. So let $x_2 = 3x - d_1$. Now $d_2 = \lfloor 3x_2 \rfloor$, and we can let $x_3 = 3x_2 - d_2$ and keep going.

As with decimal notation, base 3 representations can repeat. This happens with fractions whose denominator is not a power of 3. Here are some examples. The line over a group of digits indicates that they repeat.

$$\frac{1}{2} = \left\langle 0.\overline{1} \right\rangle_3 \quad \frac{4}{5} = \left\langle 0.\overline{2101} \right\rangle_3 \quad \frac{1}{10} = \left\langle 0.00\overline{2200} \right\rangle_3 \quad \frac{1}{7} = \left\langle 0.0\overline{102120} \right\rangle_3$$

Recall that in base 10, we can represent a terminating decimal with a series of repeating 9s: $\frac{1}{2} = 0.5 = 0.4\overline{9}$ The same happens in base 3: $\frac{1}{3} = \langle 0.1 \rangle_3 = \left\langle 0.0\overline{2} \right\rangle_3$.

A number is in C if it can be represented in base 3 using only digits 0 and 2. Here's why: At the first step, all numbers of the form $\langle 0.1 \ldots \rangle_3$ are removed to form C_1. At the second step, all remaining numbers of the form $\langle 0.01 \ldots \rangle_3$ or $\langle 0.21 \ldots \rangle_3$ are removed to form C_2. And so on. You have to use the $\left\langle \ldots \overline{2} \right\rangle_3$ representation for numbers on the right end of an interval of C_k. For example, the representation $\frac{1}{3} = \left\langle 0.0\overline{2} \right\rangle_3$ means it's $\frac{1}{3} \in C$.

Solve these problems:

1. Convert the number $\frac{7}{9}$ to base 3. $\langle 0.21 \rangle_3$
2. Convert the number $\frac{4}{11}$ to base 3. $\left\langle 0.\overline{10021} \right\rangle_3$
3. Determine whether these numbers are in C. Answer “yes” or “no”.

$\frac{9}{10}$ $= \langle 0.\overline{2200} \rangle_3$ YES $\frac{4}{15}$ $= \langle 0.\overline{2101} \rangle_3$ NO $\frac{1}{25}$ $= \langle 0.001\ldots \rangle_3$ NO $\frac{12}{13}$ $= \langle 0.\overline{220} \rangle_3$ YES

There are still a lot of points in C. In fact, there are as many points in C as there are in the whole interval $[0,1]$! Here’s why: Given $x \in C$, consider the digits of $x = \langle 0.d_1 d_2 d_3 \ldots \rangle_3$, and every d_j is either 0 or 2. Now map that to the number $f(x) = \langle 0.b_1 b_2 b_3 \ldots \rangle_2$ where

$$b_j = \begin{cases} 0 & \text{if } d_j = 0 \\ 1 & \text{if } d_j = 2 \end{cases}$$

and yes, that’s a base 2 series for $f(x)$. Note that f gives a perfect one-to-one correspondence between numbers in C and numbers in $[0,1]$, so in that sense, there are as many numbers in C as in the whole unit interval, despite the fact that C has a total length of 0.

More problems:

1. Find $f\left(\frac{1}{10}\right)$ and express it as an exact fraction in ordinary decimal notation.

$$f\left(\frac{1}{10}\right) = \boxed{\tfrac{1}{5}}$$

2. The number t is in C:

$$t = \left\langle 0.\overline{20200222} \right\rangle_3$$

Express t and $f(t)$ as exact fractions in ordinary decimal notation.

$$t = \boxed{\frac{2443}{3280}} \qquad f(t) = \boxed{\frac{167}{255}}$$