COLLEGE of CHARLESTON

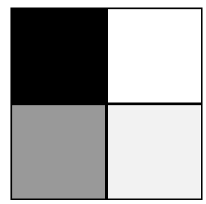
MATH MEET 2020

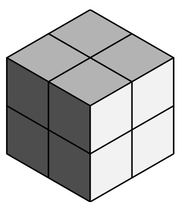
Fractal Dimension Sprint Instructions

The dimension of a shape tells you something about how it scales.

A line segment has dimension 1. If you double it, the length is multiplied by two: $L(2x) = L(x)\cdot 2$

A square has dimension 2. If you double the side length, the double-sized square consists of four unit-sized squares. The area is multiplied by the second power of two: $A(2x) = A(x) \cdot 2^2$



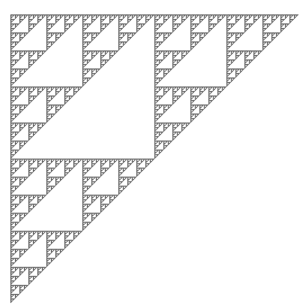


A cube has dimension 3. If you double the side length, the resulting double-sized cube consists of eight unit-sized cubes. That is, the volume is multiplied by the third power of two: $V(2x) = V(x) \cdot 2^3$

In general, the length, area, and volume of ordinary shapes have the form $M(x) = ax^d$ where x is some length that sets the object's scale, d is the dimension, and a is a constant. If you double the size of such a shape, its measure is multiplied by 2^d. For example, the volume of a sphere is $V(x) = \frac{4\pi}{3}x^3$ where x is the radius, and the power 3 on the x agrees with our intuition that spheres are three dimensional. If you double the radius, the volume increases by a factor of 8: $V(2x) = \frac{4\pi}{3}(2x)^3 = \frac{4\pi}{3}x^3 \cdot 2^3$

But there are strange and beautiful sets known as *fractals* that apparently have a fractional dimension. Like the line segment, square, and cube, they are self-similar, meaning that you can put several copies of such a shape together to form a larger one.

A famous example is the Sierpinski triangle. Three small ones can be put together to make one large one. Let x be the length of the top. Assume that its measure has the form $T(x) = ax^d$.
A double-sized one is made of three unit-sized ones, so $T(2x) = 3T(x)$.

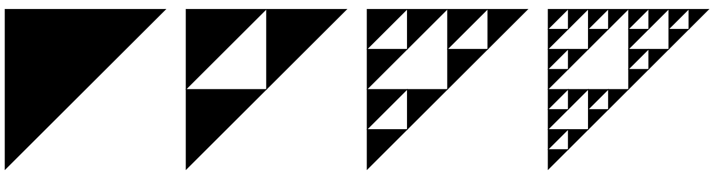


From that equation, we can solve for the dimension:

$$
\begin{aligned}
a(2x)^d &= 3ax^d \\
&\quad \text{cancel } a \text{ and } x^d \ldots \\
2^d &= 3 \\
&\quad \text{apply the natural logarithm} \ldots \\
d \ln 2 &= \ln 3 \\
d &= \frac{\ln 3}{\ln 2} \approx 1.584962501 \text{ wow!}
\end{aligned}
$$

Note that if you try this with the square, you'll get $d = 2$, and with the cube, you'll get $d = 3$.

Here's some further intuition on why the dimension of the Sierpinski triangle should be less than two. Start with a triangle of area $\frac{1}{2}$, then remove the central triangle of area $\frac{1}{8}$, then remove the three triangles around it with area $\frac{1}{32}$ each, then remove the nine triangles around all that with area $\frac{1}{128}$ each, and so on.



Thus the total remaining area is given by a geometric series calculation,

$$A = \frac{1}{2} - \frac{1}{8} - \frac{3}{8 \times 4} - \frac{3^2}{8 \times 4^2} \cdots = \frac{1}{2} - \frac{1}{8} \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k = \frac{1}{2} - \frac{1}{8} \times \frac{1}{1 - \frac{3}{4}} = 0$$

So even though the Sierpinski triangle is a planar figure, it isn't quite two dimensional because its area is zero!

On the next page, you'll find several other fractals whose dimension can be calculated the same way. Your job as a team is to calculate them!