Math Marathon

Instructions

- The problems are to be worked out individually and independently. Only textbooks and library sources may be used. Calculators and computers may be used. Each entry must be signed by a math teacher within the school to certify that all rules have been followed. Any number of entries from a school may be submitted.
- Work must be shown neatly and concisely. Explain how you got your answer. It is possible that several entries will have correct solutions, so work will be judged on exposition, clarity of thought and ingenuity, as well as correctness. The date of submission will also be considered. Electronic submissions will be accepted only once.
- Include a cover sheet for each entry with the following information: Student Name, Math Marathon, Home Address, E-mail Address, School; Year of Graduation, School Address, Signature of a Math Teacher for Verification.
- All entrants must be students who have not graduated from high school. All entrants must be registered for the Math Meet.
- The judges' decisions will be final.
- All papers are to be submitted electronically to `mathmeet@cofc.edu` or mailed to the following address

  Math Meet (Marathon)
  Department of Mathematics
  College of Charleston
  66 George Street
  Charleston, SC 29424

- All entries must be received or postmarked by February 12, 2021.

Problems

1. A right triangle has three integer sides, one of which is 2021 units. Find the minimum perimeter such a triangle could have.

2. An arithmetic progression of positive integers has sum 2021. Find the smallest possible value the largest term of such a progression could have.

3. The rational function

$$f(x)=\frac{a\,x+b}{c\,x+d}$$

is its own inverse, and the integers a, b, c, d form an arithmetic progression. What is the range of f?

4. A rectangular piece of paper is folded so that two of its diagonally opposite corners meet. The length of the fold is equal to the length of the paper. Find the ratio of the length to the width.

5. A random number generator generates two numbers a, b uniformly and independently in the interval $[0, 10]$. What is the probability that the line with equation $a\,x + b\,y = 1$ intersects two adjacent sides of the unit square $\{x, y : 0 \le x, y \le 1\}$?