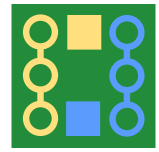
Pell's equation

Math Meet 2026



Introduction

Given a positive integer a, the following equation for unknown positive integers x and y is known as the *positive Pell equation*:

$$x^2 - ay^2 = 1 \tag{1}$$

and this is known as the *negative Pell equation*:

$$x^2 - ay^2 = -1 \tag{2}$$

For example, an easy solution with $a = 2$ is $3^2 - 2 \cdot 2^2 = 1$. Solutions to Pell equations for a particular a were important before calculators were invented because they give good rational approximations to square-roots. That is,

$$\left(\frac{x}{y}\right)^2 \approx a \pm \frac{1}{y^2} \tag{3}$$

So, from $3^2 - 2 \cdot 2^2 = 1$, we get that $\sqrt{2} \approx \frac{3}{2} = 1.5$ which is not far off from the actual $\sqrt{2} = 1.414\ldots$

The emblem at the top of the page is a representation of $3^2 = 2 \cdot 2^2 + 1$. Recall that a sum of odd numbers is a perfect square:

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2 \tag{4}$$

So $2^2 = 1 + 3$, which is represented by one yellow square followed by three yellow circles, and by one blue square followed by three blue circles. The empty space in the middle is the extra 1.

The tee-shirt is a representation of $17^2 = 2 \cdot 12^2 + 1$, with one copy of 12^2 shown in yellow as a sum of odd numbers, the second shown in blue, and the $\cdots + 1$ shown by the empty space in the middle. Numerically, $\frac{17}{12} \approx 1.417$ which is an even better approximation to $\sqrt{2}$.

How would one find the 17 and 12 other than by guessing? Use continued fractions! To begin, notice that $z^2 = 2$ can be re-written as $z^2 - 1 = 1$. Now factor the left-hand side into $(z - 1)(z + 1) = 1$ and divide, which yields

$$z = 1 + \frac{1}{1 + z} \tag{5}$$

Substitute the above equation (5) in for z on the right-hand side to construct a fraction two layers deep that is equal to z:

$$z = 1 + \frac{1}{2 + \frac{1}{1+z}} \tag{6}$$

If we keep doing this, we get a never-ending continued fraction,

$$z = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}} \tag{7}$$

If we truncate a continued fraction after a few steps, we get a *convergent*, that is, a rational number close to its actual value. For the case of $\sqrt{2}$, we get

$$\begin{aligned} 1 + \frac{1}{2} &= \frac{3}{2} = 1.5 \\ 1 + \frac{1}{2 + \frac{1}{2}} &= \frac{7}{5} = 1.4 \\ 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} &= \frac{17}{12} \approx 1.417 \end{aligned} \tag{8}$$

Notice that the numerator and denominator of each convergent solve a Pell equation:

$$\begin{aligned} 3^2 - 2 \cdot 2^2 &= 1 \\ 7^2 - 2 \cdot 5^2 &= -1 \\ 17^2 - 2 \cdot 12^2 &= 1 \end{aligned} \tag{9}$$

Questions

1. What are the next two convergents to $\sqrt{2}$? Give your answers as improper fractions $\frac{x}{y}$.
2. What solutions to Pell equations come from these convergents? Write your answers as equations like those above labeled (9).

Some number theory

Consider what happens if we take the integers $\mathbb{Z}$, throw in one extra number, specifically $\sqrt{2}$, and form the obvious set of numbers. We end up with a set named $\mathbb{Z}\left[\sqrt{2}\right]$ of numbers of the form

$$\mathbb{Z}\left[\sqrt{2}\right] = \left\{x + y \cdot \sqrt{2} \;\middle|\; x, y \in \mathbb{Z}\right\} \tag{10}$$

We can add and multiply numbers of this form, and those operations have all their usual properties. In addition we can talk about the *algebraic conjugate* of such a number,

$$\left(x + y \cdot \sqrt{2}\right)^* = x - y \cdot \sqrt{2} \tag{11}$$

and a sort of generalized length called the *modulus* of a number,

$$\left\|x + y \cdot \sqrt{2}\right\| = x^2 - 2y^2 \tag{12}$$

In general, for $w, z \in \mathbb{Z}\left[\sqrt{2}\right]$,

$$\begin{aligned} \|z\| &= z^* z = \left\|z^*\right\| \\ \|w \cdot z\| &= \|w\| \cdot \|z\| \\ \text{but } \|w + z\| &\neq \|w\| + \|z\| \end{aligned} \tag{13}$$

So solutions to the Pell equations for $a = 2$ are elements of $\mathbb{Z}\left[\sqrt{2}\right]$ with modulus ± 1. Such numbers are called *units*. Since the modulus is multiplicative, the product of any two units is another unit. That means that one or two solutions to Pell's equation can be turned into many more…

Questions

Consider the Pell equations with $a = 2$. Let $w = 3 + 2\sqrt{2}$ and $z = 7 + 5\sqrt{2}$.

3. What solutions to the *positive* Pell equation can be formed by using w^2, w^3, and w^4? Give your answer as equations as in (9).
4. Find the first three solutions to the *negative* Pell equation corresponding to powers of z, starting with z^1. Give your answer as equations as in (9).
5. Find a number u of the form $w^m z^n$ that gives the solution $41^2 - 2 \cdot 29^2 = -1$, with positive integer exponents m and n. Give your answer as an equation $u = w^{\square} z^{\square}$.
6. Find a number v of the form $w^m z^n$ that gives the solution $239^2 - 2 \cdot 169^2 = -1$, with positive integer exponents m and n. Give your answer as an equation $v = w^{\square} z^{\square}$.