College of Charleston
Math Meet 2020
Written Test – Level 3

1. A sequence of numbers $a_0, a_1, a_2, \ldots$ satisfies the **recurrence relation**

$$a_{n+2} = a_{n+1} + 2a_n \text{ for all } n \geq 0.$$

The same sequence also satisfies

$$a_{n+2} = -3a_{n+1} - 2a_n \text{ for all } n \geq 0.$$

Which of the following must also be true for all $n \geq 0$?

(A) $a_{n+1} = a_n$
(B) $a_{n+3} = a_n$
(C) $a_{n+2} = 5a_{n+1} - 4a_n$
(D) $a_{n+2} = 6a_{n+1} + 7a_n$
(E) $a_{n+3} = a_{n+2} + 4a_{n+1} - 4a_n$

2. Due to variability in wind speed and direction, when Mary stands at the origin and throws her hat into the air, it will land at one of the 9 points in Figure 1, each with probability 1/9.
If she throws her hat into the air, walks to where it lands, and throws it into the air again, what's the probability that it will then land at one of the 4 points in Figure 2?

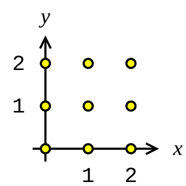


Figure 1

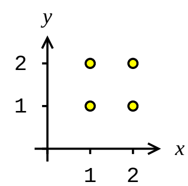


Figure 2

(A) 4/81 (B) 9/81 (C) 16/81 (D) 25/81 (E) 36/81

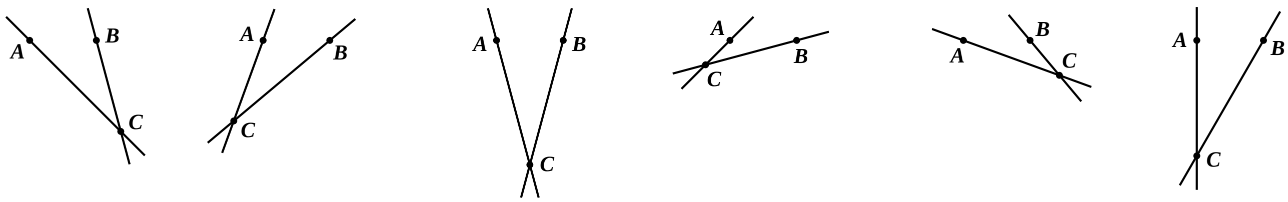
3. How many solutions are there to

$$5\sin(2x) - 8\cos x + 5\sin x = 4$$

in the interval $[0, 8\pi/3]$?

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

4. One line passes through the point $A = (-1, 0)$ in the xy-plane, and another passes through $B = (1, 0)$. The two lines intersect at a point C below the x-axis.
If $\angle ACB$ is a 30° angle, find the length of the curve consisting of all possible locations of C.



(A) $\dfrac{12\pi}{3}$ inches (B) $\dfrac{5\pi}{2}$ inches (C) $\dfrac{11\pi}{3}$ inches (D) $\dfrac{7\pi}{2}$ inches (E) $\dfrac{10\pi}{3}$ inches

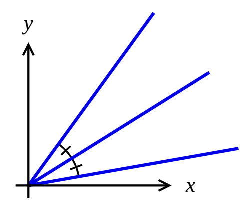
5. A red candle is 1 inch longer than a blue candle. The red candle is lit at 4:30 and the blue candle is lit at 6:00, and at 8:30 they were the same length. The blue candle burned out at 10:00 and the red candle burned out at 10:30.
Assuming that, as each candle burns, its length decreases at a constant rate, find the original length of the blue candle.

(A) 2 inches (B) 5 inches (C) 8 inches (D) 11 inches (E) 14 inches

6. Three rays with positive slope in the xy-plane meet at the origin. Their slopes are

$$m_1 < m_2 < m_3,$$

and the angle from the first to the second ray is the same as the angle from the second to the third ray (arranged according to slope). See the figure for one possible configuration of the three rays.



Which of the following could be the three slopes, (m_1, m_2, m_3)?

(A) $\left(\dfrac{1}{2}, 1, \dfrac{3}{2}\right)$ (B) $(1, 2, 3)$ (C) $(1, 2, 6)$

(D) $\left(\dfrac{3}{2}, 3, \dfrac{39}{2}\right)$ (E) $\left(\dfrac{1}{4}, \dfrac{5}{4}, \dfrac{169}{4}\right)$

7. If C is the graph of a degree 9 polynomial, and ℓ is a line, what is the largest possible number of points at which C might be tangent to ℓ?

(A) 3 (B) 4 (C) 6

(D) 8 (E) None of these

8. If $\dfrac{dy}{dx} + 5y(x) = f(x)$ and $z(x) = e^{5x}y(x)$, what can we say about z?

(A) $\dfrac{dz}{dx} = e^{-5x}f(x)$ (B) $\dfrac{dz}{dx} = e^{5x}f(x)$

(C) $\dfrac{dz}{dx} - 5z(x) = f(x)$ (D) $\dfrac{dz}{dx} = 5z(x)f(x)$

(E) None of these must be true

9. You have three boxes of unknown weight and a scale which only works for items that weigh more than 100 pounds. Since the boxes each weigh less than 100 pounds alone, you decide to weigh them together in pairs. The three different pairs weigh 113, 116 and 117 pounds respectively. By how many pounds does the weight of the heaviest box exceed the weight of the next heaviest box?

   (A) 1 pound
   (B) 2 pounds
   (C) 3 pounds
   (D) 4 pounds
   (E) cannot tell from information given

10. Let ρ be a rotation about the origin by 51° clockwise. What is the minimal number of times one must repeat ρ before every point returns to where it started?

   (A) 30
   (B) 60
   (C) 120
   (D) 360
   (E) none of these

11. When I write a certain number in base n, it looks like 25. When I write *twice* that number in base n, it looks like 52. Which of these sets *contains* the number n?

   (A) $\{6, 11, 20\}$
   (B) $\{7, 10, 14\}$
   (C) $\{12, 13, 18\}$
   (D) $\{8, 15, 19\}$
   (E) $\{5, 9, 16\}$

12. If α, β and γ are the angles of a triangle then what is

$$\cot\alpha\cot\beta + \cot\alpha\cot\gamma + \cot\beta\cot\gamma?$$

   (A) 0
   (B) 1
   (C) 2
   (D) 3
   (E) π

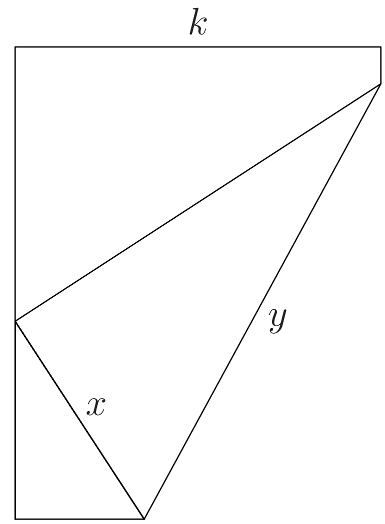
13. Evanium is a radioactive substance which decays exponentially with time. In 2 hours, a sample of Evanium will decay to 90% of its original mass. How many hours will it take for a sample to decay to 10% of its original mass?

   (A) 18
   (B) $\dfrac{2}{1-\log 9}$
   (C) $\dfrac{\ln 10}{\ln 10 - \ln 9}$
   (D) $2\ln\left(\dfrac{1}{9}\right)$
   (E) $-\log_{9/10} 10$

14. Six balls numbered 1 through 6 are randomly dropped into six boxes numbered 1 through 6. That is, one ball is dropped in each box, with an equal probability that any given ball ends up in any given box. We say that a ball is in "the right box" if it has the same number as the box that it is in. Which of these events has the *lowest* probability?

   (A) no ball ends up in the right box
   (B) exactly one ball ends up in the right box
   (C) exactly three balls end up in the right boxes
   (D) exactly five balls end up in the right boxes
   (E) all of the balls end up in the right boxes

15. The lower corner of a page of width k is folded over so as just to reach the inner edge of the page. Assuming the length of the page is great enough to allow this, find the width x of the part folded over for which the area of the triangle folded over is minimized.



(A) $\frac{k}{\sqrt{2}}$ (B) $\frac{k}{2}$ (C) $\frac{2k}{3}$ (D) $\frac{\sqrt{3}k}{2}$ (E) $\frac{\sqrt{3}k}{4}$

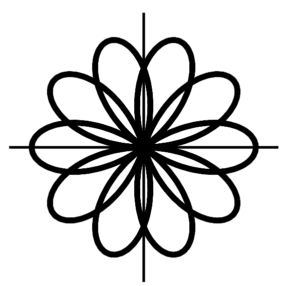
16. Ned says, "A year from now, I will be 4 times as old as my sister was when my brother was born. A year after that, my brother will be 4 times as old as I was when my sister was born. That year, my sister will be 7 times as old as our dog Frieda." All the ages are positive integers. How old is Ned's brother?

(A) Not enough information is available. (B) 6

(C) 18 (D) 30

(E) Ned is lying.

17. Which of the polar equations is graphed in the accompanying figure?



(A) $r = \cos 1.5\theta$ (B) $r = \cos 2.5\theta$ (C) $r = \cos 3.5\theta$
(D) $r = \cos 4.5\theta$ (E) $r = \cos 5.5\theta$

18. Which of these is a solution to the equation

$$2^{2x} - 8 \cdot 2^x = -12?$$

(A) $1 + \frac{\log 3}{\log 2}$ (B) $\frac{1}{2}\log 6$ (C) $1 + \log\frac{3}{2}$
(D) $\log 3$ (E) none of these

19. A is the set $\{1, 2, 3, 4\}$. B and C are nonempty sets. $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7\}$ and $A \cap B \cap C = \emptyset$. If the number of elements of B is greater than the number of elements in C, find the smallest possible sum of the elements of B.

(A) 6 (B) 8 (C) 10 (D) 11 (E) 13

20. Find all solutions to the equation $x^6 + 3x^5 + 3x^4 + 6x^3 + 3x^2 + 3x + 1 = 0$ given that i is a double root.

(A) $(-3 \pm \sqrt{5})/2,\ i,\ -i$ (B) $(-3 \pm \sqrt{5})/2,\ i,\ -i,\ 1,\ -1$

(C) $(3 \pm \sqrt{5})/2,\ i,\ -i$ (D) $(3 \pm \sqrt{5})/2,\ i,\ -i,\ 1,\ -1$

(E) $(-3 \pm \sqrt{5})/2,\ i,\ 1$

21. In the complex plane, there is a straight line that contains all six roots of $z^6 - 8iz^4 - 19z^2 + 12i = 0$. Which of the following points is on that line?

(A) $1 + i$ (B) 1 (C) i

(D) $1 + 2i$ (E) None of these

22. A sphere is placed on a horizontal plane during a sunny day. At a certain instant of time, its shadow reaches 10 meters from the point where the sphere touches the plane. At the same time, a 1 meter tall post casts a 2 meter long shadow. What is the radius of the sphere, expressed in meters?

(A) $\frac{5}{2}$ (B) $9 - 4\sqrt{5}$ (C) $10\sqrt{5} - 20$ (D) $8\sqrt{10} - 23$ (E) $6 - \sqrt{15}$

23. In a given forest, suppose there are more trees than there are leaves on any *single* tree. Which of the following must be true?

I. There exist two trees with a leaf in common.
II. There exist at least two trees which have the same number of leaves.
III. There exists a tree with no leaves.

(A) I only (B) II only (C) III only (D) either II or III (E) either I or III

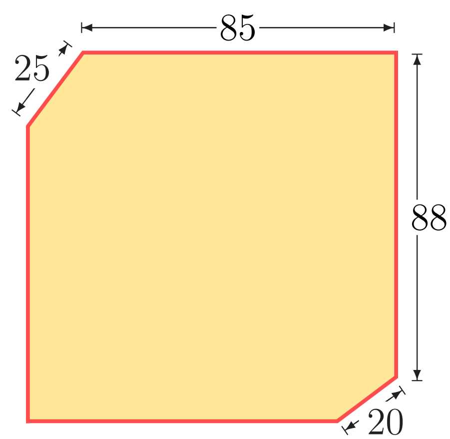
24. Find the intersection of all intervals having the form

$$\left(1 - \frac{1}{n},\ 5 + \frac{4}{n}\right),$$

where n is a positive integer.

(A) $(0, 6)$ (B) $[1, 6]$ (C) $[1, 5]$ (D) $(1, 5)$ (E) $[0, 9]$

25. Two corners were cut off a 100×100 square resulting in the hexagon shown in the figure. Four of the sides have the lengths indicated by the labels. What is the total perimeter of this shape?



(A) 368 (B) 372 (C) 378 (D) 382 (E) 388

2020 Answers / Level 3 Test

1. D
2. D
3. B
4. E
5. C
6. E
7. B
8. B
9. C
10. C
11. D
12. B
13. B
14. D
15. C
16. C
17. B
18. A
19. B
20. A
21. A
22. C
23. D
24. C
25. D