College of Charleston
Math Meet 2021
Written Test – Level 3

1. A leaky oil pipeline beneath a lake causes a circular slick of uniform thickness to form on the surface. If the thickness of the slick is constant over time, if r denotes the radius of the slick and t denotes the number of minutes since the slick began to form, and if the oil leaks into the water at a constant rate (measured in liters per minute), which of these is true?

   (A) dr/dt is constant.

   (B) dr/dt is inversely proportional to r.

   (C) dr/dt is inversely proportional to $\sqrt{r}$.

   (D) dr/dt is proportional to t.

   (E) dr/dt is proportional to $\sqrt{t}$.

2. During his best show, a magician pulled 26 rabbits out of his 5 different hats. We know that
   - he pulled out at least 1 rabbit from the first hat;
   - for each other hat he pulled out more rabbits than from the previous one;
   - he pulled out 4 times as many rabbits from the last hat as from the second hat.

   How many rabbits did the magician pull out of the third hat?

   (A) 3 (B) 4

   (C) 5 (D) 6

   (E) The given information is contradictory

3. A regular triangle with sides of length 3 is inscribed into a circle. A second circle is inscribed between the triangle and the first circle. Find the radius of the second circle.

   (A) $\frac{\sqrt{3}}{4}$ (B) $\frac{3}{4}$ (C) $\frac{\sqrt{3}}{2}$

   (D) $\frac{3}{2}$ (E) None of these

4. Ewa has a flight of 9 stairs to climb. If each step she takes can cover either one or two stairs, how many ways are there for her to reach the 9th stair?

   (A) 38 (B) 45 (C) 46 (D) 55 (E) 64

5. Red played White in a soccer match yesterday. Before the game, there were some predictions:
   - The game will not end in a tie.
   - Red will score against White.
   - Red will win.
   - Red will not lose.
   - Exactly three goals will be scored in the game.

   Exactly three of these turned out to be true. What was the final score?

   (A) Red 2, White 1

   (B) Red 1, White 2

   (C) Red 3, White 0

   (D) Red 0, White 3

   (E) Not enough information was given to determine the score.

6. David tosses a fair coin until it comes up heads and records the number of tosses it takes him. Lacking a coin, Emmy takes a fair six-sided die, labels two sides "heads" and the other four sides "tails," and then tosses the die until it comes up heads. What is the probability that David's number of coin tosses equals Emmy's number of die rolls?

(A) $1/2$ (B) $1/4$ (C) $2/9$ (D) $1/5$ (E) $3/11$

7. A beaker is full of an r% alcohol solution. An identical beaker is 1/3 full of an s% alcohol solution. Two thirds of the full beaker is poured into the 1/3-full beaker and the resulting solution is mixed thoroughly. How many more times must this pouring-and-mixing procedure be repeated so that the difference between the concentrations in the two beakers is less than $0.002|r-s|$?

(A) 2 (B) 3

(C) 5 (D) 8

(E) None of these is the correct answer.

8. The parabola $y=-x^2$ is slid in the plane in such a way that it remains tangent to $y=x^2$ and its axis of symmetry remains vertical. What curve does its vertex trace out?

(A) $-4y=x^2$ (B) $y=\frac{1}{3}x^2$ (C) $4y=x^2$ (D) $2y=x^2$ (E) $2x=y^2$

9. Mingus catches some fish. He gives the three largest fish to his dog, thus reducing the total weight of his catch by 35%. He then gives the three smallest fish to his cat, reducing the remaining total weight by 5/13. He ate the remaining fish for dinner. How many fish did Mingus catch?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

10. Suppose $p(x)$ is a polynomial satisfying

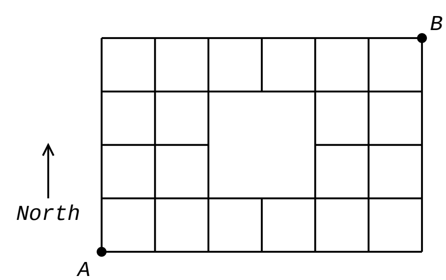
$$p(x+1)-p(x-1)-2p'(x)=0$$

for all numbers x. Find $p(4)$ if $p(-2)=4$ and $p(0)=-2$ and $p(2)=0$.

(A) 7 (B) 8 (C) 9

(D) 10 (E) Not enough information

11. An Uber must travel from A to B on the map in the figure. If it never travels south or west, how many different routes can it take?
In the correct answer, the ones digit is . . .

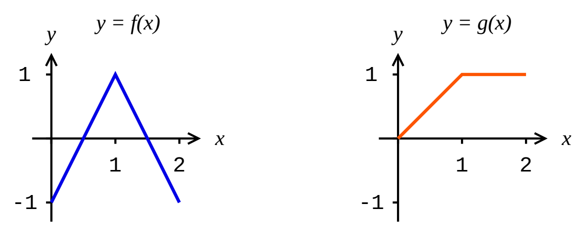


(A) 0 (B) 1 (C) 2 (D) 4 (E) 8

12. Let $f(x)$ be the piecewise-linear function shown in the figure, and extend $f(x)$ to $(-\infty,\infty)$ so as to have period 2.
Part of the graph of a function

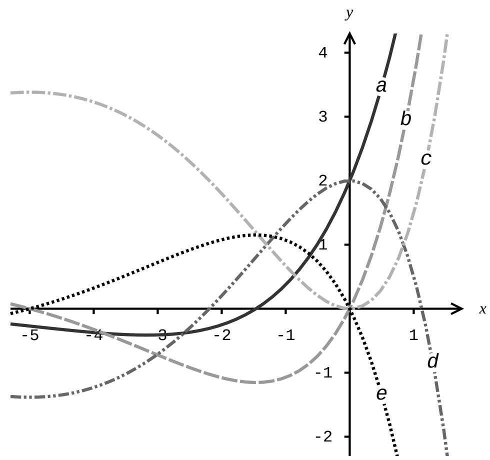
$$g(x)=af(x)+bf(\tfrac{1}{2}x)+cf(\tfrac{1}{4}x)$$

appears in the figure. Find $g(3)$.



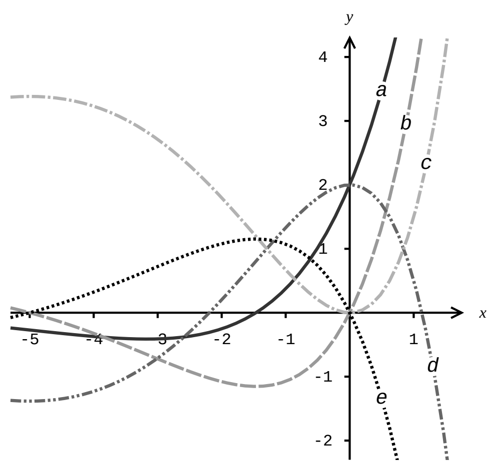
(A) 3/2 (B) 1/2 (C) 0 (D) -1/2 (E) -2

13. The accompanying figure shows the graphs of a function $f(x)$, its derivative $f'(x)$, its second derivative $f''(x)$, and two other functions. Find the graph of $f'(x)$.



(A) a (B) b (C) c (D) d (E) e

14. The accompanying figure shows the graphs of a function $f(x)$, its derivative $f'(x)$, its second derivative $f''(x)$, and two other functions. Find the graph of $f(x)$.



(A) a (B) b (C) c (D) d (E) e

15. Let f be a function such that for all x, $f(2+x) = -f(2-x)$ and $f(-2+x) = -f(-2-x)$. Which of the following must also be true?

(A) $f(x) = f(x+8)$ (B) $f(x) = f(x+4)$

(C) $f(x) = -f(x+4)$ (D) f is constant

(E) None of the others must be true

16. The degree-four polynomial $y = p(x)$ has three distinct critical points, (x_1, y_1), (x_2, y_2), (x_3, y_3), in some order, and we know that $y_1 = y_2$. Which of the following must be true?

(A) $y_3 = y_1$ (B) $y_3 = 2y_1$ (C) $x_1 + x_2 = 2x_3$

(D) $x_3 = x_1 + x_2$ (E) None of these

17. The complex number z is reflected through the line $y = x$. What is the result?

(A) $\frac{i}{z}$ (B) $-\bar{z}$ (C) iz

(D) $i\bar{z}$ (E) None of these

18. Let $p(x)$ be a degree 7 polynomial, all of whose coefficients are integers. If $p(2) = 0$ and $p(3) = 0$, what can we say about $p(5)$?

(A) It's equal to 0

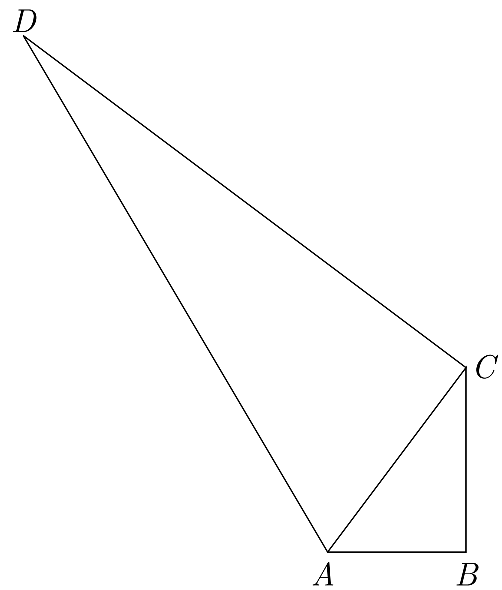
(B) It's divisible by 6

(C) It's greater than 0

(D) It's odd

(E) There isn't enough information to say anything about $p(5)$

19. Referring to the figure, the lengths of the sides are as follows: AB is 3, BC is 4, AC is 5, CD is 12, and AD is 13. What is the distance from D to the (extended) line AB?



(A) 10 (B) $\frac{56}{5}$ (C) $8\sqrt{2}$

(D) $\frac{21}{2}$ (E) None of these

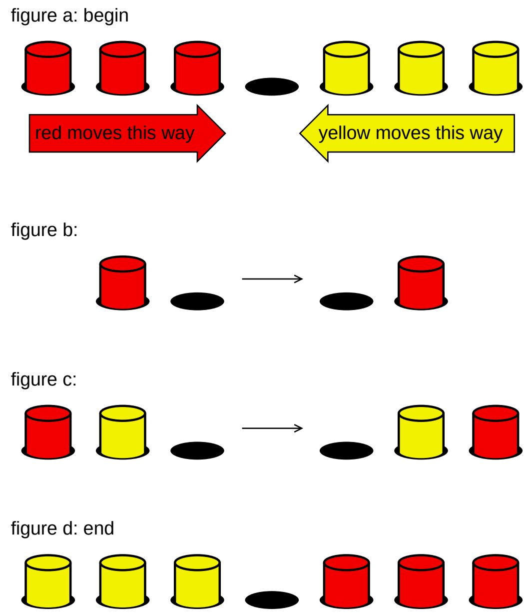
20. For $n \geq 2$, let $a_n = 6a_{n-1} - 9a_{n-2}$ and let $a_0 = \frac{1}{2020}$ and $a_1 = \frac{1}{2021}$. For $n \geq 1$, define $b_n = \frac{a_n}{3^n} - \frac{a_{n-1}}{3^{n-1}}$. What can we say about b_n?

(A) $b_n = 3b_{n-1}$ (B) $b_n = -3b_{n-1}$

(C) $b_n = b_{n-1}$ (D) $b_n = 0$

(E) None of the others must be true

21. A game begins with 3 red and 3 yellow pegs placed in 7 holes as shown in figure a.
Pegs may move forward only. The forward direction for red pegs is right; the forward direction for yellow is left.
Two types of moves are permitted: a peg can move one step forward into an empty hole, or jump a peg of the opposing color into an empty hole on the other side (illustrated with red moving in figures b and c).
The object of the game is to move pegs, not necessarily alternating between the two colors, until all the yellow pegs are on the left and all the red on the right, as in figure d. In order to achieve this, if red moves first, what colors must move 4th and 5th?



(A) red 4th and 5th.

(B) red 4th, yellow 5th

(C) yellow 4th, red 5th

(D) yellow 4th and 5th.

(E) There is more than one possible answer.

22. A ray of light originates at $(-3, 4)$ in the x-y plane, reflects off of line ℓ at $(0, 0)$ and ends up at $(24, 7)$. What is the equation of ℓ?

(A) $y = -2x$ (B) $y = -x$ (C) $y = -x\sqrt{2}$
(D) $3y = -x$ (E) None of these

23. When performing long division on two polynomials $p(x)$ and $q(x)$, a student chose not to stop when she reached the remainder and instead wrote

$$\frac{p(x)}{q(x)} = 2x + 1 - 3x^{-1} + 2x^{-2} + x^{-3} - 3x^{-4} + 2x^{-5} + x^{-6} - 3x^{-7} + 2x^{-8} + \cdots$$

If $q(1) = 3$, find $p(1)$.

(A) 5 (B) 3 (C) 0 (D) -3 (E) -5

24. Suppose that f and g are continuous functions, defined for all real numbers, and that for all x, $f(x) \geq g(x)$. Suppose that the area between the graphs of f and g and the vertical lines $x = 3$ and $x = 5$ is 4. Define $h(x) = f(x) + 7$ and $k(x) = g(x) + 5$. What is the area between the graphs of h and k and the vertical lines $x = 3$ and $x = 5$?

(A) 4

(B) 6

(C) 0

(D) 8

(E) Cannot be determined from the given information

25. "Sonny is 7 years older than Max, but in 5 years, Max will be exactly half Sonny's age." Which of these equations models this problem?

(A) $x + 7 = 2(x + 5)$ (B) $\frac{1}{2}(x + 12) = (x + 5)$ (C) $x + 5 = 2(x + 7)$

(D) $x + 7 = \frac{1}{2}(x - 5)$ (E) $\frac{1}{2}(x + 7) = (x + 5)$

2021 Answers / Level 3 Test

1. B
2. B
3. A
4. D
5. B
6. B
7. C
8. D
9. C
10. D
11. A
12. D
13. B
14. C
15. A
16. C
17. D
18. B
19. B
20. C
21. A
22. D
23. A
24. D
25. B