The Area of a Regular Pentagon
All-Day Sprint / Math Meet 2010

Note: *This Sprint should be done without the use of calculators, computers or other references. Use only your brains and your writing implements.*

If you've studied trigonometry, I bet you know the exact formulas for the sine and cosine of $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{6}$. These are easy enough to figure out by cutting a square in half and by cutting an equilateral triangle in half. But what about something like $\frac{\pi}{5}$? It turns out that with a little help from complex numbers, you can also get an exact formula for the sine and cosine of $\frac{\pi}{5}$ . . . And from that you can find an exact algebraic formula for the area of a pentagon.

First, something you may not yet have seen is that there is a nifty relationship between trig functions, the natural exponential function e^x, and complex numbers. Using $i = \sqrt{-1}$, the basic formulas are

$$\cos\omega = \frac{e^{i\omega} + e^{-i\omega}}{2} \qquad \sin\omega = \frac{e^{i\omega} - e^{-i\omega}}{2i} \qquad e^{i\omega} = \cos\omega + i\sin\omega$$

So for example, you can rediscover the double angle identities like so:

$$\left(e^{i\theta} + e^{-i\theta}\right)^2 = 2^2(\cos\theta)^2$$

But you can also expand it out

$$\left(e^{i\theta} + e^{-i\theta}\right)^2 = e^{2i\theta} + 2 + e^{-2i\theta} = \left(e^{2i\theta} + e^{-2i\theta}\right) + 2$$

Then use the basic formulas with $\omega = 2\theta$ to get

$$\left(e^{i\theta} + e^{-i\theta}\right)^2 = 2\cos 2\theta + 2$$

Now set the two calculations equal

$$2^2(\cos\theta)^2 = 2\cos 2\theta + 2$$

and you can solve for $\cos 2\theta$ in terms of $\cos\theta$ to get the familiar

$$\cos 2\theta = 2(\cos\theta)^2 - 1$$

* Find a formula for $\cos 3\theta$ as a polynomial in $\cos\theta$. That is, find an identity of the form

$$\cos 3\theta = A_0 + A_1 \cos\theta + A_2(\cos\theta)^2 + A_3(\cos\theta)^3$$

* Find a formula for $\cos 5\theta$ as a polynomial in $\cos\theta$.

* Plug $\theta = \frac{\pi}{10}$ into your $\cos 5\theta$ identity to form a 5th degree polynomial equation satisfied by the unknown quantity $x = \cos\frac{\pi}{10}$.

* Solve that polynomial to find an exact formula for $\cos\frac{\pi}{10}$ in terms of addition, subtraction, multiplication, division, and square roots. Be clever!

* Find an exact formula for the area of a regular pentagon inscribed in a circle of radius 1 in terms of addition, subtraction, multiplication, division, and square roots.