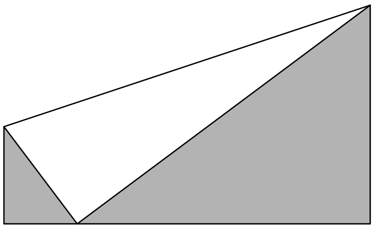
Pythagorean Paper Sprint

All-Day Sprint
Math Meet 2007
College of Charleston

You can a make a collection of three right triangles such as the one shown in the figure from any rectangular piece of paper. Hold the paper so that it is wider than it is tall, and fold the top-left corner down until it just touches the bottom edge. Then flatten it out so that the crease is straight. Pythagoras would love it, don't you think?



But, not every piece of paper you start with will produce triangles with nice measurements!

You see, I really like *whole* numbers. I started with a piece of paper whose length and width (measured in centimeters) were both two-digit, whole numbers. Moreover, when I folded it I found that the triangle on the right was a perfect 3-4-5 right triangle. (That is, 5 times the length of one leg is the same as 3 times the length of the hypotenuse and 5 times the length of the other leg is 4 times the length of the hypotenuse.) Finally, I was so pleased when I computed the area of the two shaded triangles (in square centimeters) and found that they *also* were two-digit, whole numbers!

The question you must answer is this: What were the areas of the two shaded triangles (in square centimeters)?

Answer: The triangle areas are 96 and 24 square centimeters!

Here's the explanation. Call the width x and the height y. Also, call the length from the intersection on the base to the bottom right corner X and the length from the bottom left corner to the vertical location of the crease Y.

Note first that the bottom right triangle is a right triangle with legs X and y and hypotenuse x. Since y and the area $Xy/2$ are integers, X must be an integer, too. Then, since we know that it is a $3-4-5$ triangle, there is a number S so that either

$$I) \qquad x = 5S,\ X = 4S,\ y = 3S$$

or

$$II) \qquad x = 5S,\ X = 3S,\ y = 4S$$

for some integer S.

We can find limits on S based on the fact that the area of this triangle is a two digit integer, but it depends on whether the triangle is of form (I) or (II). If (I) is true then the area is $3S^2$ and so $2 \leq S \leq 5$. On the other hand, if (II) is true then the area is $6S^2$ and so $2 \leq S \leq 4$.

Now, we need to consider the other triangle to determine the answer more specifically. It is important to recognize that the triangles are similar. To see that this is true, look at the three angles formed at the point where the corner intersects the bottom edge. There is an angle on the left that is part of the left triangle, a right angle and an angle in the right triangle. Together, they add up to $180°$. But, since the left triangle also has a right angle, its third angle must also be the same size as the third angle above and similarly the third angle of the right triangle is the same as the angle on the left of the intersection point. Thus

$$\frac{Y}{X} = \frac{x - X}{y} \Rightarrow Y = (x - X)X/y.$$

In particular, we conclude from this that the area of the left triangle is

$$\frac{(x - X)^2 X}{2y}.$$

If (I) is true then the area is $2S^2/3$. For this to be an two-digit integer S must be a multiple of 3 which is at least 9. But, that is not possible since we already determined that S could be no bigger than 5. There is no way that the right triangle could be in the form (I) then.

Alternatively, if (II) is true then the area of the left triangle is $3S^2/2$. For this to be a two-digit integer, S would have to be an even number which is at least 4. We earlier determined that S could not be any bigger than 4. But, $S = 4$ does work!

In conclusion, the only way it could turn out as described in the question is if (I) is true with $S = 4$. In that case $x = 20$ and $y = 16$. More importantly, the areas of the two triangles are $3S^2/2 = 24$ and $6S^2 = 96$.